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A Game-Theoretic Model of Pollution Control with Asymmetric Time Horizons*

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Abstract In the contribution a problem of pollution control is studied within the game-theoretic framework (Kostyunin et al., 2013; Gromova and Plekhanova, 2015; Shevkoplyas and Kostyunin, 2011). Each player is assumed to have certain equipment whose functioning is related to pollution control. The i-th player's equipment may undergo an abrupt failure at time T_i. The game lasts until any of the players' equipment breaks down. Thus, the game duration is defined as $T = \min(T_1, \ldots, T_n)$, where T_i is the time instant at which the i-th player stops the game.
We assume that the time instant of the i-th equipment failure is described by the Weibull distribution. According to Weibull distribution form parameter, we consider different scenarios of equipment exploitation, where each of player can be in "an infant", "an adult" or "an aged" stage. The cooperative 2-player game with different scenarios is studied.

Keywords: differential game, cooperative game, pollution control, random duration, Weibull distribution.

1. Introduction

When considering game-theoretic problems of pollution control it is important to take into account the fact that the game may end abruptly. The reason for this can be an equipment failure, an economical break-down or a natural disaster among many others. In this paper we consider one particular case when the game duration is determined by the life duration of the equipment. Typically, when describing the life circle of a technical system one considers three different stages: the "infant" stage, the "adult" or regular stage, and the "aged" or weared-out stage. It is well known that the life-time for all these stages can be well described by the Weibull distribution (Weibull, 1951).

In this paper, we consider a pollution control problem for n players. We assume that the equipment of each player at the beginning of the game can be in any of three states ("infant", "adult" or "aged"). Thus the life-time of the equipment differs for each player. The game ends with the occurence of the first failure.

The proposed approach is illustrated by an example of pollution control problem with two players.

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2. The Problem Statement

Consider a game-theoretic model of pollution control based on the models (Breton et al., 2005; Shevkoplyas and Kostyunin, 2011). There are n players (countries) involved in the game. Each player i manages his emission $e_i \in [0, b_i]$, $b_i > 0$, $i = \overline{1,n}$. Each country is assumed to have certain equipment whose functioning is related to pollution control.

The game starts at the time instant t_0. The i-th player's equipment may undergo an abrupt failure at time T_i. The game lasts until any of the players' equipment breaks down. Thus, the game duration is defined as $T = \min(T_1, \dots, T_n)$, where T_i is the time instant at which the i-th player stops the game.

We assume that the time instant of the i-th equipment failure is the random variable T_i with known probability distribution function $F_i(t)$, $\quad i = \overline{1,n}$ (Petrosjan and Murzov, 1966). Assume also that $\{T_i\}_{i=1}^n$ – independent random variables. It is obvious, that $T = min\{T_1, T_2, \dots, T_n\}$ is a random variable too. Using the cumulative distribution functions of the random variables $\{T_i\}_{i=1}^n$, we can write the expression for $F(t)$.

Proposition 1. *Let $\{T_i\}_{i=1}^n$ – independent random variables, with probability distribution functions $\{F_i(t)\}_{i=1}^n$. Then probability distribution function $F(t)$ of the random variable $T = min\{T_1, T_2, \dots, T_n\}$ has the following form:*

$$F(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t)) . \qquad (1)$$

Proof. According to the distribution function definition:

$$F(t) = P\{T < t\} .$$

Here

$$P\{T < t\} = 1 - P\{T \geq t\} .$$

The random variable T is defined as $T = min\{T_1, T_2, \dots, T_n\}$, so:

$$P\{T \geq t\} = P\{min\{T_1, T_2, \dots, T_n\} \geq t\} .$$

$\{T_i\}_{i=1}^n$ – independent random variables, then

$$P\{min\{T_1, T_2, \dots, T_n\} \geq t\} = P\{T_1 \geq t\} P\{T_2 \geq t\} \dots P\{T_n \geq t\} .$$

Using again the distribution function definition, we have

$$P\{T_1 \geq t\} P\{T_2 \geq t\} \dots P\{T_n \geq t\} = (1 - F_1(t))(1 - F_2(t)) \dots \ (1 - F_n(t)) ,$$

i.e.:

$$F(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t)) .$$

□

The net revenue of player i at time instant t is given by quadratic functional form:

$$R_i\left(e_i\right) = e_i(t)\left(b_i - \frac{1}{2}e_i(t)\right),\ t \in [t_0, T], \tag{2}$$

$$b_i > 0,\ i = \overline{1, n}. \tag{3}$$

Denote the stock of accumulated net emissions by P(t). The dynamics of the stock is given by the following equation with initial condition:

$$\dot{P}(t) = \sum_{i=1}^{n} e_i(t),\ t \in [t_0, T], \tag{4}$$

$$P(t_0) = P_0. \tag{5}$$

The expected integral payoff of the player i can be represented as the following mathematical expectation:

$$K_i\left(P_0, t_0, e_1, e_2, \ldots, e_n\right) = E\left(\int_{t_0}^{T}\big(R_i(e_i) - d_i P(s)\big)ds\right),$$

where $d_i P(t)$ – is a cost of player i for decreasing of his emission at the moment t.

Then we have the following integral payoff for player i:

$$K_i\left(P_0, t_0, e_1, e_2, \ldots, e_n\right) = \int_{t_0}^{\infty}\int_{t_0}^{t}\big(R_i(e_i) - d_i P(s)\big)ds dF(t). \tag{6}$$

After simplification of the integral payoff (Kostyunin and Shevkoplyas, 2011), we get

$$K_i\left(P_0, t_0, e_1, e_2, \ldots, e_n\right) = \int_{t_0}^{\infty}\big(R_i(e_i) - d_i P(s)\big)\big(1 - F(s)\big)ds. \tag{7}$$

Denote the described game starting at the time instant t_0 from the situation P_0 by $\Gamma(t_0, P_0)$. Let the game $\Gamma(t_0, P_0)$ develops along the trajectory $P(t)$. Then at the each time instant $\theta \in [t_0; T]$ players enter new game (subgame) $\Gamma(\theta, P(\theta)))$ with initial state $P(\theta)$ and duration $(T - \theta)$. The expected payoff of the player i under the condition that the game is not finished before the moment θ can be calculated by following formula:

$$K_i\left(P(\theta), \theta, e_1, e_2, \ldots, e_n\right) = \frac{1}{1 - F(\theta)}\int_{\theta}^{\infty}\big(R_i(e_i) - d_i P(s)\big)\big(1 - F(s)\big)ds. \tag{8}$$

Further we assume an existence of a density function:

$$f(t) = F^{'}(t), \tag{9}$$

and using the Hazard function $\lambda(t)$ which is given by the following definition:

$$\lambda(t) = \frac{f(t)}{1 - F(t)}, \tag{10}$$

we have

$$1 - F(s) = e^{-\int\limits_0^s \lambda(\tau)d\tau}. \tag{11}$$

We can prove the following proposition.

Proposition 2. *Let* $\{T_i\}_{i=1}^n$ *– independent random variables with probability distribution function* $F_i(t)$, $i = \overline{1, n}$ *and the hazard functions* $\{\lambda_i(t)\}_{i=1}^n$. *Then for the random variable* $T = min\{T_1, T_2, \dots, T_n\}$ *the hazard function* $\lambda(t)$ *can be calculated by the following formula:*

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t). \tag{12}$$

Proof. As we say in (1):

$$F(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t)), \tag{13}$$

using (11), we have:

$$\prod_{i=1}^{n} (1 - F_i(t)) = e^{-\int\limits_0^t \lambda(\tau) d\tau}. \tag{14}$$

Taking the log of both sides in (14), we obtain:

$$\ln\big(\prod_{i=1}^{n} (1 - F_i(t))\big) = -\int\limits_0^t \lambda(\tau)\, d\tau. \tag{15}$$

Then we have:

$$\sum_{i=1}^{n} \ln\big(1 - F_i(t)\big) = -\int\limits_0^t \lambda(\tau)\, d\tau. \tag{16}$$

Similarly, we see that

$$1 - F_i(t) = e^{-\int\limits_0^t \lambda_i(\tau) d\tau}, \tag{17}$$

and

$$\ln\big(1 - F_i(t)\big) = -\int\limits_0^t \lambda_i(\tau)\, d\tau. \tag{18}$$

Substituting (18) in (16), we obtain:

$$\sum_{i=1}^{n} \int\limits_0^t \lambda_i(\tau)\, d\tau = \int\limits_0^t \lambda(\tau)\, d\tau. \tag{19}$$

So we can conclude that

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t).$$

□

One of probability distributions that can be used for description of random variables T_i is Weibull Law. The Weibull failure rate function is given by:

$$\lambda(t) = \lambda\delta t^{\delta-1}, \quad t > 0;\ \lambda > 0;\ \delta > 0, \tag{20}$$

where δ is the shape parameter and λ is the scale parameter of the distribution. Using Weibull distribution allows to consider three "scenarios" of the game in the sense of behaviour of the random variables T_i:

1. $\delta < 1$ corresponds to "burn-in" period, when the equipment failure is mostly caused by deficiencies in design (new equipment);
2. $\delta = 1$ corresponds to "adult" period, when failures are due to random events;
3. $\delta > 1$ corresponds to "wear-out" period (worn-out equipment).

For Weibull distribution we have:

$$1 - F_i(s) = e^{-\int\limits_0^s \lambda_i \delta_i \tau^{\delta_i - 1} d\tau} = e^{-\lambda_i s^{\delta_i}}. \tag{21}$$

Then the payoff of player i in subgame $\Gamma(\theta, P(\theta))$ can be represented as:

$$K_i\left(P(\theta), \theta, e_1, e_2, \ldots, e_n\right) = e^{\sum\limits_{i=1}^n \lambda_i \theta^{\delta_i}} \int_\theta^\infty \big(R_i(e_i) - d_i P(s)\big) e^{-\sum\limits_{i=1}^n \lambda_i s^{\delta_i}} ds. \tag{22}$$

Suppose that players are agree to cooperate and maximize the joint payoff:

$$K_1 + K_2 + \cdots + K_n = \frac{1}{1 - F(\Theta)} \int_\theta^\infty \big(R_1(e_1) + R_2(e_2) + \cdots + R_n(e_n) - \\ - (d_1 + d_2 + \cdots + d_n) P(s)\big)\big(1 - F(s)\big) ds. \tag{23}$$

According to the Proposition 1:

$$F(s) = 1 - \prod_{i=1}^n \left(1 - F_i(s)\right),$$

then:

$$1 - F(s) = e^{-(\lambda_1 s^{\delta_1} + \lambda_2 s^{\delta_2} + \cdots + \lambda_n s^{\delta_n})}. \tag{24}$$

As a result, the expected total payoff of players in the subgame $\Gamma(\theta, P(\theta))$ with initial state $P(\theta)$ and duration $(T - \theta)$ is given by the following equation:

$$\sum_{i=1}^n K_i = e^{\lambda_1 \theta^{\delta_1} + \lambda_2 \theta^{\delta_2} + \cdots + \lambda_n \theta^{\delta_n}} \int_\theta^\infty \big(R_1(e_1) + R_2(e_2) + \cdots + R_n(e_n) - \\ -(d_1 + d_2 + \cdots + d_n) P(s)\big) e^{-(\lambda_1 s^{\delta_1} + \lambda_2 s^{\delta_2} + \cdots + \lambda_n s^{\delta_n})} ds. \tag{25}$$

3. 2-player game

Consider 2-player cooperative game-theoretic model of pollution control.

Let T_1 – the time instant of the equipment failure for the player 1 with probability distribution function $F_1(t)$ and failure rate function $\lambda_1(t)$. T_2 – the time instant of the equipment failure for the player 2 with probability distribution function $F_2(t)$ and failure rate function $\lambda_2(t)$.

The game duration is defined as $T = min\{T_1, T_2\}$.

The game is considered over time $t \in [0, T]$, where T is a random variable with known probability distribution function $F(t) = 1 - (1 - F_1(t))(1 - F_2(t))$ and failure rate function $\lambda(t) = \lambda_1(t) + \lambda_2(t)$.

We use Weibull distribution as a distribution of random variables T_1 and T_2. The players are assumed to have the identical scale parameter $\lambda = \lambda_1 = \lambda_2$.

Using (25), we get the equation for the joint payoff of players:

$$K_1 + K_2 = \int_0^{\infty} \big(R_1(e_1) + R_2(e_2) - (d_1 + d_2)P(s)\big)e^{-\lambda(s^{\delta_1}+s^{\delta_2})}ds. \quad (26)$$

To find the optimal emissions $\overline{e}_1$, $\overline{e}_2$ for players 1, 2, we apply Pontrygins maximum principle.:

$$\max_{\substack{e_1\in[0,b_1],\\ e_2\in[0,b_2]}} (K_1 + K_2) = \int_0^{\infty} \big(R_1(\overline{e}_1) + R_2(\overline{e}_2) - (d_1 + d_2)P(s)\big)e^{-\lambda(s^{\delta_1}+s^{\delta_2})}ds, \quad (27)$$

where

$$\begin{aligned} K_i =& \; K_i\left(P_0, e_1, e_2\right), \quad i = 1, 2, \\ \dot{P}\left(t\right) =& \; e_1\left(t\right) + e_2\left(t\right), \\ P\left(0\right) =& \; P_0. \end{aligned}$$

The Hamiltonian for this problem is as follows:

$$\begin{aligned} &H(P, e_1, e_2, \Lambda) = \\ &= \left(e_1(t)(b_1 - \frac{1}{2}e_1(t)) + e_2(t)(b_2 - \frac{1}{2}e_2(t)) - (d_1 + d_2)P(t)\right) e^{-\lambda\left(t^{\delta_1}+t^{\delta_2}\right)} + \\ &\qquad + \Lambda\left(t\right)\left(e_1(t) + e_2(t)\right). \quad (28) \end{aligned}$$

From the first-order optimality condition

$$\frac{\partial H}{\partial e_i} = \left(b_i - e_i\right) e^{-\lambda\left(t^{\delta_1}+t^{\delta_2}\right)} + \Lambda(t) = 0, \; i = 1, 2, \quad (29)$$

we get the following formulas for optimal emissions:

$$\overline{e}_i(t) = b_i + \Lambda(t)e^{\lambda\left(t^{\delta_1}+t^{\delta_2}\right)}, \; i = 1, 2. \quad (30)$$

Adjoint variable $\Lambda(t)$ can be found from the from the differential equation:

$$\dot{\Lambda} = -\frac{\partial H}{\partial P}. \quad (31)$$

Then

$$\Lambda(t) = (d_1 + d_2) \int_0^t e^{-\lambda\left(s^{\delta_1}+s^{\delta_2}\right)}ds + c. \quad (32)$$

We consider the problem with time $t \in [0, \infty)$ and the condition for $\Lambda(t)$ has a form:

$$\lim_{t\to\infty} \Lambda(t) = 0. \quad (33)$$

3.1. Optimal emissions

Different scenarios with possible conditions of players' equipment are considered in this section.

The normal operating mode of the equipment Let's consider the case, when the equipments of both players are used in the normal operating mode. It means that $\delta_1 = \delta_2 = 1$ ("adult" scenario).

Using (30), we have the following form for optimal emissions

$$\overline{e}_i(t) = b_i + \Lambda(t)e^{2\lambda t}. \tag{34}$$

Adjoint variable in this case has a form:

$$\Lambda(t) = (d_1 + d_2)\int\limits_0^t e^{-2\lambda s}ds + c = -\frac{(d_1 + d_2)}{2\lambda}e^{-2\lambda t} + \frac{(d_1 + d_2)}{2\lambda} + c, \tag{35}$$

where c can be found from (33):

$$c = -\frac{(d_1 + d_2)}{2\lambda}. \tag{36}$$

So

$$\Lambda(t) = -\frac{(d_1 + d_2)}{2\lambda}e^{-2\lambda t}. \tag{37}$$

Then substituting (37) in (34) we obtain the two optimal strategies:

$$\overline{e}_i(t) = \begin{cases} 0, & b_i \leq \dfrac{d_1 + d_2}{2\lambda}; \\ b_i - \dfrac{d_1 + d_2}{2\lambda}, & b_i > \dfrac{d_1 + d_2}{2\lambda}, i = 1, 2. \end{cases} \tag{38}$$

The mode of normal operation of the equipment and worn-out equipment Assume now that the equipment of the first country is in the normal operating mode ($\delta_1 = 1$) and the second one uses the worn-out equipment ($\delta_2 > 1$). Without loss of generality we assume $\delta_2 = 2$ (the Rayleigh distribution).

Using (30), we have the following form for optimal emissions:

$$\overline{e}_i(t) = b_i + \Lambda(t)e^{\lambda(t+t^2)}. \tag{39}$$

Adjoint variable in this case has a form:

$$\begin{aligned} \Lambda(t) = (d_1 + d_2)\int\limits_0^t e^{-\lambda(s+s^2)}ds + c = \\ = \frac{(d_1 + d_2)\sqrt{\pi}e^{\frac{1}{4}\lambda}}{2\sqrt{\lambda}}\left(\text{erf}(\sqrt{\lambda}t + \frac{1}{2}\sqrt{\lambda}) - \text{erf}(\frac{1}{2}\sqrt{\lambda})\right) + c, \end{aligned} \tag{40}$$

where c can be found from (33). Then we get

$$\Lambda(t) = \frac{(d_1 + d_2)\sqrt{\pi}e^{\frac{1}{4}\lambda}}{2\sqrt{\lambda}}\left(\text{erf}(\sqrt{\lambda}t + \frac{1}{2}\sqrt{\lambda}) - 1\right). \tag{41}$$

Denote by

$$\widehat{e}_i(t) = b_i + \frac{(d_1 + d_2)\sqrt{\pi}e^{\frac{1}{4}\lambda}e^{\lambda(t+t^2)}}{2\sqrt{\lambda}}\left(\text{erf}(\sqrt{\lambda}t + \frac{1}{2}\sqrt{\lambda}) - 1\right),$$

and

$$A_{\widehat{e}_i} = \{t \mid \widehat{e}_i(t) < 0\}.$$

Then we get:

$$\overline{e}_i(t) = \begin{cases} 0, & \text{if } t \in A_{\widehat{e}_i}; \\ \widehat{e}_i(t), & \text{otherwise.} \end{cases} \tag{42}$$

The new equipment for both countries Assume that both of countries use the new equipment, the shape parameters in this case are $\delta_1 = \delta_2 = \frac{1}{2}$.

Using (30), (32),(33), we have:

$$\overline{e}_i(t) = b_i + \Lambda(t)e^{2\lambda\sqrt{t}}, \tag{43}$$

$$\Lambda(t) = -\frac{(d_1+d_2)(2\lambda\sqrt{t}e^{-2\lambda\sqrt{t}} + e^{-2\lambda\sqrt{t}} - 1)}{2\lambda^2} + c, \tag{44}$$

where

$$c = -\frac{(d_1+d_2)}{2\lambda^2}. \tag{45}$$

Then we get

$$\Lambda(t) = -\frac{(d_1+d_2)(2\lambda\sqrt{t}e^{-2\lambda\sqrt{t}} + e^{-2\lambda\sqrt{t}})}{2\lambda^2}. \tag{46}$$

Let's find optimal emissions in this case. If $b_i \leq \frac{d_1+d_2}{2\lambda^2}$, then

$$\overline{e}_i(t) = 0. \tag{47}$$

If $b_i > \frac{d_1+d_2}{2\lambda^2}$, then

$$\overline{e}_i(t) = \begin{cases} 0, & t \geq \left(\frac{2\lambda^2 b_i - d_1 - d_2}{2\lambda(d_1+d_2)}\right)^2; \\ b_i - \frac{(d_1+d_2)(2\lambda\sqrt{t}+1)}{2\lambda^2}, & 0 \leq t < \left(\frac{2\lambda^2 b_i - d_1 - d_2}{2\lambda(d_1+d_2)}\right)^2. \end{cases} \tag{48}$$

The worn-out equipment for both countries Consider the problem in the case when equipment of both countries is worn-out ($\delta_1 > 1$, $\delta_2 > 1$). Fix the following values of the shape parameters: $\delta_1 = \delta_2 = 2$.

Using (30), (32),(33), we have:

$$\overline{e}_i(t) = b_i + \Lambda(t)e^{2\lambda t^2}, \tag{49}$$

$$\Lambda(t) = \frac{(d_1+d_2)\sqrt{2\pi}\,\mathrm{erf}(\sqrt{2\lambda}t)}{4\sqrt{\lambda}} + c, \tag{50}$$

where c can be found from (33).

Then we get the following form for adjoint variable:

$$\Lambda(t) = \frac{(d_1+d_2)\sqrt{2\pi}(\mathrm{erf}(\sqrt{2\lambda}t) - 1)}{4\sqrt{\lambda}}. \tag{51}$$

Denote by

$$\widetilde{e}_i(t) = b_i + \frac{(d_1+d_2)\sqrt{2\pi}e^{2\lambda t^2}(\mathrm{erf}(\sqrt{2\lambda}t)-1)}{4\sqrt{\lambda}},$$

and

$$A_{\widetilde{e}_i} = \{t \mid \ \widetilde{e}_i(t) < 0\}.$$

Then we get:

$$\overline{e}_i(t) = \begin{cases} 0, & \text{if } \ t \in A_{\widetilde{e}_i}; \\ \widetilde{e}_i(t), & \text{otherwise.} \end{cases} \tag{52}$$

The graphic representation of the optimal emissions for four scenarios of the game we can see at the Fig. 1.

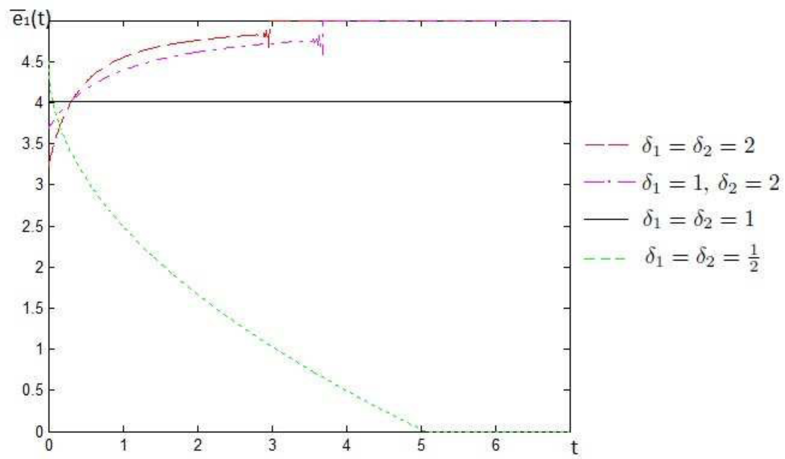

Fig. 1: Optimal emissions of the player 1 for four scenarios of the game.

4. Conclusion

In the paper a problem of pollution control was studied within the game-theoretic framework. Each player was assumed to have certain equipment whose functioning is related to pollution control. The i-th player's equipment may undergo an abrupt failure at random time T_i which is described by the Weibull distribution with different parameters corresponding to different modes of operation of the equipment. The game lasts until any of the players' equipment breaks down. Thus, the game duration is defined as $T = \min(T_1, \ldots, T_n)$, where T_i is the time instant at which the i-th player stops the game.

A cooperative 2-player game with different scenarios was studied in detail.

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