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Coordination in Multilevel Supply Chain*

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Abstract There is a task of coordination in the multilevel supply chains with the tree-like structure taking into consideration the linearity of supply in the final markets that is discussed in this article. Three ways are suggested by authors in order to solve the chain coordination problem, i. e. to the rule of the players strategies choice that are satisfying the certain criteria of optimality. The first way is a decentralized solution that will be issued only when all the supply chain participants act independently from each other. The second way is the optimization of the overall chains revenue in the cooperative game, so called centralized solution. Finally, the third solution is the Nash weighted solution that is created by the optimization of the Nash weighted multiplication. Based on the particular example there is a comparison of all the ways discussed in the article.

Keywords: Multilevel supply chains, tree-like structure, overall chains revenue, Nash weighted solution.

1. Introduction

Modern world is closely connected with trade and business, which supply chain is the indispensable part of. The necessity of firms to sell their goods after being produced make them develop their trading activities by systems of trade flows and trade connections organization. Every year because of the progress and globalization pressure there is a growth of not only the number of these systems, but also of the difficulty, namely their structure and scale. In addition, there are appearing problems of optimization in the already organized supply chains, however the importance of their solution might be sometimes underestimated. As a result, badly organized operational performance leads to the loss and nonnetted gain. Therefore, not only the supply chains wide incidence, but also importance of the optimization solutions under the revenue criteria makes the problem of coordination among the players in the supply chain could not more up-to-date. In terms of this, the goal of the current article is the elaboration of the participants coordination way that is aimed to optimize supply chain under the revenue criteria.

In the following article one of the most omni-purpose and widespread kind of supply chains is examined, namely, the multilevel supply chains with the tree-like

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structure (the example of such a chain is depicted on the Fig. 1). The problem for these chains coordination is not well studied, because the supply chains modeling of that particular structure has just recently begun. This problem was examined in the works by Corbett C., Karmarkar U. S. (2001) and Carr M. S., Karmarkar U. S. (2005) for the first time. However, later on modeling of the multilevel supply chains was continued in the direction of pricing contracts and horizontal competition (Kaya, 2012; Cho, 2014). Only recently scientists have returned back to the optimization of the multilevel supply chains (Zhou et al., 2015).

In the following paper, there are three approaches to the coordination of participants, that are based on the different models of interaction or on the optimization criteria. For each of the approaches we are describing the process of the participants interaction, based on that we are formulating the optimization criteria and are designing the satisfying way of solutions design.

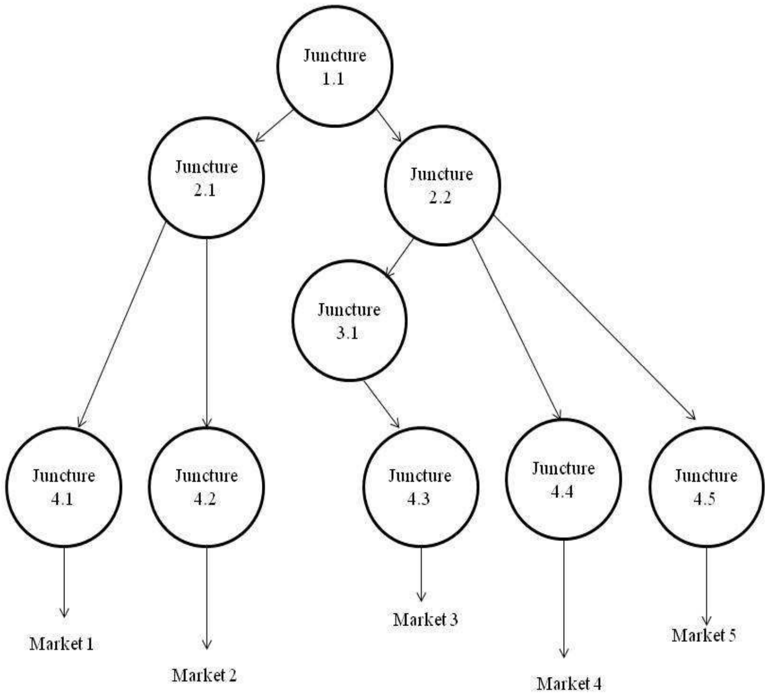

Fig. 1. The example of the multilevel supply chain with tree-like distribution structure

The further structure of the article will be organized in the following way: Section 2 is devoted to the mathematical formalization of multilevel supply chains with the tree-like distribution structure; in the Sections 3, 4, and 5 there are decentralized, centralized and weighted Nash solutions that are analyzed; in the Section 6 there is an example stated and the delivered results compared; in the final, 7th, section there is a summary and the results of the research presented.

2. Mathematical formalization of the supply chains

Let us look at the tree-like graph $G = (X, F)$ with a mutual peaks of X and a mutual verges of F. The root peak of this tree can be named as x^1. In the set of peaks X let us define the sets of $X_1, \dots, X_l, X_i \subset X$ in the following way:

$$\begin{aligned} &X_1 = \{x^1\}, X_l = \{x \in X | F_x = \emptyset\}, \\ &X_{k+1} = (F_x \setminus X_l) \text{ for } x \in X_k, k = \overline{1, l-2}, \text{ if } (F_x \setminus X_l) = \emptyset \end{aligned} \quad (1)$$

Comment 1. The inserted multitudes are setting the division of multitude X, such as $U^l_{i=1} X_i = X, X_e \bigcap X_r = \emptyset, e \neq r$.

Definition 1. Subset of junctures $X_i \subset X, i = 1, \dots, l$, will be named as the set of peaks (junctures) of the Level i. The junctures from the set of X_l will be named the final or the finite.

We will denote the junctures x from the multitude X as x^i_j, where the upper index is equal to the number of the level X_i, where this peak is situated and the lower index to the order number of this peak in the multitude X_i. For the uniformity, the root juncture x^1 will be denoted as x^1_1. What is more by m_i we will understand the number of the junctures of the level of I, i.e. $m_i = |X_i|$, where $|X_i|$ - the power of the multitude X_i.

Definition 2. We will say that dissection of $X_1, \dots, X_l$ the multitude of X peaks, that was defined under the rule of (1), is defining the supply chain with the tree-like control (distributive) structure.

Definition 3. The sector of the peak $x^i_j \in X \setminus X_l$ is the name of the multitude $F_{x^i_j}$.

Comment 2. The multitude of the sectors together with the root peak are controlling the dissection on the multitudes of peaks X.

Under the multitude S^i_j we will understand the multitude of pairs of indexes of these tech junctures that are included in the sector of the juncture $x^i_j \in X \setminus X_l$, so as $S^i_j = \{(k, h) | x^k_h \in F_{x^i_j}\}$. Let us notice that under the generation $S^i_j \neq \emptyset$.

Assume that every peak $x^i_j, i = \overline{1, l}, j = \overline{1, m_l}$, of supply chain consists of finite plurality of elements $\{x^i_{jk}\}^{n_{ij}}_{k=1}$, for which the set of lattice points is defined $\{v_{ijk}\}^{n_{ij}}_{k=1}, \forall k : v_{ijk} \geq 0$, where n_{ij} is any positive integer that is not less than 1. This plurality of elements is a context-wise a group of competitive firms that are producing and consuming the homogeneous product as well as having the different v_{ijk} production costs (the production power is meant to be unrestricted). For each firm $x^i_{jk} \in x^i_j$ let us work in the variable $q_{ijk} \geq 0$, that is characterizing the running production volume of this firm as well as the integrated volume of the homogeneous product that was produced by all firms $\{x^i_{jk}\}^{n_{ij}}_{k=1}$ from the juncture x^i_j, let us call as $Q_{ij} = \sum^{n_{ij}}_{k=1} q_{ijk}$. Then for the sector of each juncture $x^i_j \in X \setminus X_l$ supply chain the following condition is considered to be fulfilled:

$$Q_{ij} = \sum_{k=1}^{n_{ij}} q_{ijk} = \sum_{r,h:(r,h) \in S^i_j} \sum_{t=1}^{n_{rh}} q_{rht}, \quad (2)$$

meaning that there is no deficit or surplus of production in the supply chain.

For every juncture $x_j^i \in X$ let us work in the variable p_{ij} that is equivalent sense wise the price according to that firms $\{x_{jk}^i\}_{k=1}^{n_{ij}}$ from the juncture x_j^i are selling the unit of the good produced. It is considered that for the every of the final peaks $x_j^i \in X_l$ there is the following linear function prescribed

$$p_{lj} = a_{lj} - b_{lj} Q_{lj} \tag{3}$$

where $a_{lj} > 0$, $b_{lj} > 0$. In fact, it means that the final peaks are realizing their product in the non-competitive consumer markets that are functioning according to the Cournot model with the linear correspondence that could be expressed by the formula (3).

Definition 4. The set of definitions $(\{q_{ijk}\}_{i,j,k}, \{p_{ij}\}_{i,j}$ is defining the trading flow d in the supply chain.

Definition 5. Flow d will be named feasible, if $p_{lj} > 0$, $Q_{ij} > 0$, $j = \overline{1, m_l}$.

Let the set D be the multitude for all the feasible flows in the supply chain. For each of the firms $\{x_{jk}^i\}_{k=1}^{n_{ij} \in x_j^i}$ for $i = \overline{1, l}, j = \overline{1, m_l}$ let us define the function π_{ijk} – the revenue function that is set on the multitude D among all the feasible trading flows in the following way:

$$\pi_{ijk}(d) = \begin{cases} q_{11k}(p_{11} - v_{11k}), & \text{if } i = 1; \\ q_{ljk}(a_{ij} - b_{lj} Q_{lj} - p_{rh} - v_{ljk}), & \text{if } i = l; \\ q_{ijk}(p_{ij} - p_{rh} - v_{ijk}), & \text{in all other cases.} \end{cases}$$

where $p_{rh} : x_j^i \in S_h^r$.

Let us arrange the multitude of peaks X supply chain: in the first place is a root peak, then the junctures of the second level in the ascending order, then of the third, fourth levels and up to the final inclusively, i.e. we will receive the arranged system $\{x_1^1, x_1^2, x_2^2, \ldots, x_{mi}^l\}$. This arranged multitude of all the junctures (let us denote it with N) of supply chain we will consider as the multitude of players.

The multitude of U_{ijin} the strategy of the player x_j^i will be considered as the multitude of all the possible vectors $u_{ij} \in D$, where u_{ij} is created out of the arranged order of variables that are defined for all the firms $\{x_{jk}^i\}_{k=1}^{n_{ij}} \in x_j^i$ and are situated within the area defining the feasible flow, namely:

$$U_{ij} = \begin{cases} \{u_{ij} = (q_{ij1}, \ldots, q_{ijn_{ij}}, p_{ij}) \in D\}, \, x_j^i \in N; \, i = \overline{1, l-1}, \, j = \overline{1, m_l} \\ \left\{u_{lj} = (q_{lj1}, \ldots, q_{ljn_{lj}}) \in D\right\}, \quad x_j^l \in N, \, j = \overline{1, m_l}. \end{cases} \tag{4}$$

Within this article we will examine three ways of the objectives formulation and optimality criteria. Let us consider the case when each of the supply chains participants is acting independently from each other and exclusively in favor of his own interests, then such model and corresponding to it solution will be named decentralized. If all the supply chain participants are cooperating and predefining to act concordantly in order to maximize the total revenue of the supply chain, then such problem will be called centralized. The third variant weighted Nash solution is the result of the optimization problem solution, in which as a matter of the objective function the weighted Nash solution is stated whereas as a status quo point it is the solution of the decentralized model in the same supply chain that is used.

3. Game-theoretic model of the multilevel decentralized supply chain

3.1. Formalization and the optimality criteria

First of all, let us describe the procedure of the decision-making in the decentralized model:

Step 1. The root juncture is denoting the selling price for the junctures of its sector.

Step 2. The peaks of the second level in the supply chain having received the information from the root juncture, are defining the price for a good to the peaks of their sectors. Then the procedure is repeating up to the junctures of the next to last level inclusively.

Step 3. The final peaks based on the prices having received from their suppliers, and supply functions are defining the volumes of production of the good to the market.

Step 4. The procedure of volumes disposal is happening between firms on the each of the peaks of the final level.

Step 5. Information about the volumes is arriving to all the upper-situated levels and within each juncture is happening the procedure of volumes disposal between firms.

Step 6. Calculation of revenue from each participant in the supply chain.

The decision-making process that is described above characterizes the decentralized multilevel tree-like supply chain as the conflict-managed system, with the hierarchical structure, therefore these systems specifically are defined by the order of the managerial levels that are followed one by one in the order of the denoted priority.

Definition 6. The feasible flow d^* will be called optimal if it is fulfilled:

$$\pi_{ijk}(d^*)\pi_{ijk}(d^{ij}), \forall i = \overline{1,l}, j = \overline{1,m_i}, k = \overline{1,n_{ij}}, \tag{5}$$

where (d^{ij}) is the flow that was created by the deviation of the strategy u_{ij} of the player x^i_j.

Let us look into the plus-sum multistage game Γ with hierarchical structure that is revealed in a plurality $\langle Y, \{U_i\}_{i\in Y}, \{H_i\}_{i\in Y}\rangle$ where $Y = \{1, 2, \ldots, k\}$ is the multitude of players with dissection into the subsets according to the priority, U_i is the multitude of managing stimulus of the player i to the players that are subject to him, H_i is the payoff functional of the player i that was set in the Cartesian product of sets U_i leading the players $U = \prod_{i\in Y} U_i$. Control vector $u = (u_1, \cdots, u_k)$ is forming the situation in the game Γ. At the present time lets take the arranged multitude of supply chain junctures $N = \{x^1_1, \ldots, x^l_{m_l}\}$ as the multitude of the players Y, as the multitude of the controlling actions multitude U_{ij} of players strategies $x^i_j \in N$. Each of the player $x^i_j \in N$ will be assigned in the correspondence the vector $\pi^i_j = (\pi_{ij1}, \pi_{ij2}, \ldots, \pi_{ijn_{ij}})$. Then as the payoff functions of the players let us take accordingly the arranged set of vectors $\pi^i_j : \pi = \{\pi^1_1, \ldots, \pi^l_{m_l}\}$.

Then the plurality $\langle N, \{U_{ij}\}_{i,j:x^i_j\in N}, \{\pi^i_j\}_{i,j:x^i_j\in N}\rangle$ is defined as the plus-sum multistage game with the hierarchical structure, and the task of decentralized model coordination of the multilevel supply chain is the process of finding the Nash equilibrium in the multilevel hierarchical game with the complete information.

3.2. Construction of the two-level decentralized supply chain solution

Let us begin the coordination task with the particular example when $l = 2$, namely there are only 2 levels in the supply chain and it has the form of vector (see the Fig. 2).

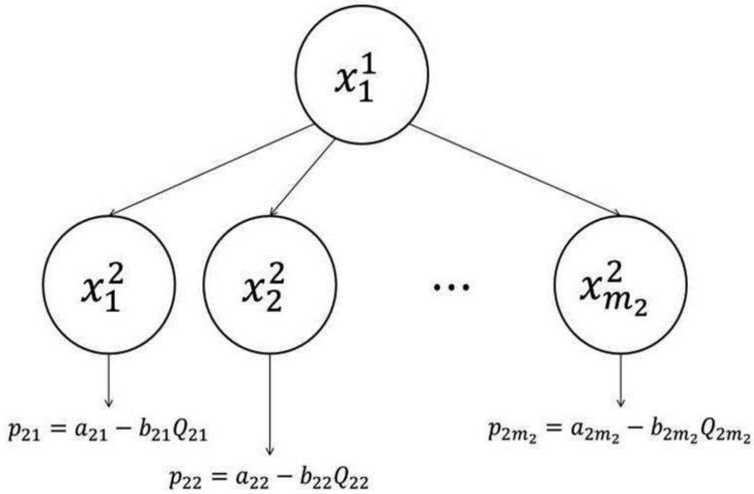

Fig. 2. Two-level supply chain.

Let us look at the firm k in the finite juncture x_j^2, where $1 \leq j \leq m_2$, $1 \leq k \leq n_{2j}$. For it the revenue formula equation looks like:

$$\pi_{2jk} = q_{2jk}(p_{2j} - p_{11} - v_{2jk}) \tag{6}$$

Let us apply in this formula the equation for p_{2j}, taking into the consideration the supply function (3), namely:

$$p_{2j} = a_{2j} - b_{2j}Q_{2j}, Q_{2j} = \sum_{k=1}^{n_{2j}} q_{2jk}.$$

Then we will get the following equation:

$$\pi_{2jk} = q_{2jk}(a_{2j} - b_{2j}\sum_{h=1}^{n_{2j}} q_{2jh} - p_{11} - v_{2jk}) \tag{7}$$

For the conforming of the assumption (5) let us apply to the revenue function (7) the condition of necessity for the maximum:

$$\frac{\partial \pi_{2kj}}{\partial q_{2jk}} = \left(a_{2j} - b_{2j}\sum_{h=1}^{n_{2j}} q_{2jh} - p_{11} - v_{2jk}\right) - b_{2j}q_{2jk} = 0,$$

and express the q_{2jk}:

$$q_{2jk} = \frac{1}{2b_{2j}}\left(a_{2j} - p_{11} - v_{2jk}\right) - \frac{1}{2}\sum_{h=1,h\neq k}^{n_{2j}} q_{2jh}. \tag{8}$$

Let us perform (6) (8) for all $k=\overline{1,n_{2j}}$ and we will come up to the system:

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix} \cdot \begin{pmatrix} q_{2j1} \\ q_{2j2} \\ \vdots \\ q_{2jn_{2j}} \end{pmatrix} = \begin{pmatrix} \frac{1}{b_{2j}}\left(a_{2j}-p_{11}-v_{2j1}\right) \\ \frac{1}{b_{2j}}\left(a_{2j}-p_{11}-v_{2j2}\right) \\ \vdots \\ \frac{1}{b_{2j}}\left(a_{2j}-p_{11}-v_{2jn_{2j}}\right) \end{pmatrix} \tag{9}$$

Matrix of the system (9) is a non-degenerate due to the linear connection of the series (columns), thus, this system may be solved in a one-valued way relatively to the all q_{2jk}.

Let us find the opposite matrix for the matrix of the system (9):

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}^{-1}_{[n_{2j}\times n_{2j}]} = \begin{pmatrix} \frac{n_{2j}}{n_{2j}+1} & \frac{-1}{n_{2j}+1} & \cdots & \frac{-1}{n_{2j}+1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{n_{2j}+1} & \frac{-1}{n_{2j}+1} & \cdots & \frac{n_{2j}}{n_{2j}+1} \end{pmatrix};$$

and let us multiply on the left-hand side both of the sides (9) by this matrix:

$$\begin{pmatrix} q_{2j1} \\ q_{2j2} \\ \vdots \\ q_{2jn_{2j}} \end{pmatrix} = \begin{pmatrix} \frac{n_{2j}}{n_{2j}+1} & \frac{-1}{n_{2j}+1} & \cdots & \frac{-1}{n_{2j}+1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{n_{2j}+1} & \frac{-1}{n_{2j}+1} & \cdots & \frac{n_{2j}}{n_{2j}+1} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{b_{2j}}\left(a_{2j}-p_{11}-v_{2j1}\right) \\ \frac{1}{b_{2j}}\left(a_{2j}-p_{11}-v_{2j2}\right) \\ \vdots \\ \frac{1}{b_{2j}}\left(a_{2j}-p_{11}-v_{2jn_{2j}}\right) \end{pmatrix} \tag{10}$$

Having accomplished the multiplication in the par (10) we will get the following equation for q_{2jk}:

$$q_{2jk} = \frac{1}{b_{2j}(n_{2j}+1)}\left(a_{2j}-p_{11}-n_{2j}v_{2jk}+\sum_{h=1,h\neq k}^{n_{2j}} v_{2jh}\right), k=\overline{1,n_{2j}}. \tag{11}$$

The found value of the variables is in reality the point of maximum to the revenue function, i.e.:

$$\frac{\partial^2 \pi_{2jk}}{\partial q_{2jk}^2} = -b_{2j}-b_{2j} = -2b_{2j} < 0,$$

remain valid b_{2j};

$$\frac{\partial^2 \pi_{2jk}}{\partial q_{2jk}\partial q_{2jr}} = 0, \forall r \neq k.$$

We can find the equation for Q_{2j}:

$$Q_{2j} = \sum_{k=1}^{n_{2j}} q_{2jk} = \sum_{k=1}^{n_{2j}} \frac{1}{b_{2j}(n_{2j}+1)}\left(a_{2j}-p_{11}-n_{2j}v_{2jk}+\sum_{h=1,h\neq k}^{n_{2j}} v_{2jh}\right) = \\ = \frac{n_{2j}(a_{2j}-p_{11})-\sum_{k=1}^{n_{2j}} v_{2jk}}{b_{2j}(n_{2j}+1)},\ j=\overline{1,m_2}. \tag{12}$$

Let us have a look into the root sector. For the firm k from the root peak 1.1 the function of revenue has the following form:

$$\pi_{11k} = q_{11k}(p_{11} - v_{11k}), k = \overline{1, n_{11}}. \tag{13}$$

The condition of surplus elimination and deficit (2) is expressed in the formula

$$Q_{11} = \sum_{k=1}^{n_{11}} q_{11k} = \sum_{j=1}^{m_2} Q_{2j} = \sum_{j=1}^{m_2} \frac{n_{2j}(a_{2j} - p_{11}) - \sum_{h=1}^{n_{2j}} v_{2jh}}{b_{2j}(n_{2j}+1)},$$

from that one can express the value p_{11} from variables q_{11k}:

$$p_{11} = \frac{-\sum\limits_{k=1}^{n_{11}} q_{11k} + \sum\limits_{j=1}^{m_2} \left(\frac{n_{2j}a_{2j} - \sum_{h=1}^{n_{2j}} v_{2jh}}{b_{2j}(n_{2j}+1)} \right)}{\sum\limits_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right)}. \tag{14}$$

Let us plug received equation (14) in the revenue formula (13):

$$\pi_{11k} = q_{11k} \left(\frac{-Q_{11} + \sum\limits_{j=1}^{m_2} \left(\frac{n_{2j}a_{2j} - \sum_{h=1}^{n_{2j}} v_{2jh}}{b_{2j}(n_{2j}+1)} \right)}{\sum\limits_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right)} - v_{11k} \right), k = \overline{1, n_{11}}, \tag{15}$$

and then let us use the maximum condition of necessity to the equation for the revenue functions (15):

$$\frac{\partial \pi_{11k}}{\partial q_{11k}} = \left(\frac{-Q_{11} + \sum\limits_{j=1}^{m_2} \left(\frac{n_{2j}a_{2j} - \sum_{h=1}^{n_{2j}} v_{2jh}}{b_{2j}(n_{2j}+1)} \right)}{\sum\limits_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right)} - v_{11k} \right) - \\ - \frac{q_{11k}}{\sum\limits_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right)} = 0, \; k = \overline{1, n_{11}}. \tag{16}$$

Having leaved the variables $q_1 1k$ in the left side and having transferred other parameters to the right side, we will receive the following system:

$$\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \begin{pmatrix} q_{111} \\ q_{112} \\ \vdots \\ q_{11n_{11}} \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_{j=1}^{m_2} \left(\frac{n_{2j} a_{2j} - \sum_{h=1}^{n_{2j}} v_2 jh}{b_{2j}(n_{2j}+1)} \right) - v_{111} \sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right) \\ \sum_{j=1}^{m_2} \left(\frac{n_{2j} a_{2j} - \sum_{h=1}^{n_{2j}} v_2 jh}{b_{2j}(n_{2j}+1)} \right) - v_{112} \sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right) \\ \vdots \\ \sum_{j=1}^{m_2} \left(\frac{n_{2j} a_{2j} - \sum_{h=1}^{n_{2j}} v_2 jh}{b_{2j}(n_{2j}+1)} \right) - v_{11n_{11}} \sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right) \end{pmatrix}. \quad (17)$$

Matrix of the system (17):

$$\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}_{[n11 \times n_{11}]}$$

is a non-degenerate due to the linear independence of its columns (rows). That is why we can express in a one-valued way the meanings of the variables q_{11k}, having multiplied this system to the opposite matrix that has the form:

$$\begin{pmatrix} \frac{n_{11}}{n_{11}+1} & \frac{-1}{n_{11}+1} & \cdots & \frac{-1}{n_{11}+1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{n_{11}+1} & \frac{-1}{n_{11}+1} & \cdots & \frac{n_{11}}{n_{11}+1} \end{pmatrix}.$$

We will receive the equations for $q_{11j}, j = \overline{1, n_1 1}$:

$$\begin{pmatrix} q_{111} \\ q_{112} \\ \vdots \\ q_{11n_{11}} \end{pmatrix} = \begin{pmatrix} \frac{n_{11}}{n_{11}+1} & \frac{-1}{n_{11}+1} & \cdots & \frac{-1}{n_{11}+1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{n_{11}+1} & \frac{-1}{n_{11}+1} & \cdots & \frac{n_{11}}{n_{11}+1} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \sum_{j=1}^{m_2} \left(\frac{n_{2j}a_{2j} - \sum_{h=1}^{n_{2j}} v_{2jh}}{b_{2j}(n_{2j}+1)} \right) - v_{111} \sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right) \\ \sum_{j=1}^{m_2} \left(\frac{n_{2j}a_{2j} - \sum_{h=1}^{n_{2j}} v_{2jh}}{b_{2j}(n_{2j}+1)} \right) - v_{112} \sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right) \\ \vdots \\ \sum_{j=1}^{m_2} \left(\frac{n_{2j}a_{2j} - \sum_{h=1}^{n_{2j}} v_{2jh}}{b_{2j}(n_{2j}+1)} \right) - v_{11n_{11}} \sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right) \end{pmatrix} \tag{18}$$

After simplification (18) we will come up to the pars:

$$q_{11k} = \frac{1}{(n_{11}+1)} \sum_{j=1}^{m_2} \frac{1}{b_{2j}} \left(n_{2j}a_{2j} - \sum_{h=1}^{n_{2j}} v_{2jh} - n_{11}v_{11k} + \sum_{r=1, r\neq k}^{n_{11}} v_{11r} \right), \quad k = \overline{1, n_{11}}. \tag{19}$$

The values found (19) are in reality the points of maximum, because

$$\begin{aligned} \frac{\partial^2 \pi_{11k}}{\partial q_{11k}^2} &= \frac{-1}{\sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right)} + \frac{-1}{\sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right)} = \\ &= \frac{-2}{\sum_{j=1}^{m_2} \left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right)} < 0, \end{aligned} \tag{20}$$

due to the fact that $\left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)} \right) > 0, \forall\, j = \overline{1, m_2}$;

$$\frac{\partial \pi_{11k}}{\partial q_{11k} \partial q_{11r}} = 0, \forall\, r \neq k.$$

In the formula (19) all the parameters are known, because they are the predefined ones in the supply chain. As a consequence, the meanings of the variables q_{11k} are known as well. Thus, further we can consequently find the meanings of the variables $p_{11}, Q_{11}, q_{2jk}, j = \overline{1, m_2}, k = \overline{1, n_{2j}}$ $p_{2j}, j = \overline{1, m_2}$. That is how the optimal flow for the two-level decentralized supply chain was found and the problem of coordination was solved. Analytical equations of the meanings of values in equilibrium are stated in the Table 1.

Table 1. Analytical equations for the meanings of variables in equilibrium

Variable	Equation
$q_{11k},\ k=\overline{1,n_{11}}$	$\frac{1}{n_{11}+1}\sum_{j=1}^{m_2}\frac{1}{b_{2j}}\left(n_{2j}a_{2j}-\sum_{h=1}^{n_{2j}}v_{2jh}- -n_{11}v_{11k}+\sum_{r=1,r\neq k}^{n_{11}}v_{11r}\right)$
Q_{11}	$\frac{1}{n_{11}+1}\sum_{j=1}^{m_2}\frac{1}{b_{2j}}\left(n_{11}n_{2j}a_{2j}- -n_{11}\sum_{h=1}^{n_{2j}}v_{2ih}-\sum_{r=1}^{n_{11}}v_{11r}\right)$
p_{11}	$\frac{-\sum_{k=1}^{n_{11}}q_{11k}+\sum_{j=1}^{m_2}\left(\frac{n_{2j}a_{2j}-\sum_{h=1}^{n_{2j}}v_{2jh}}{b_{2j}(n_{2j}+1)}\right)}{\sum_{j=1}^{m_2}\left(\frac{n_{2j}}{b_{2j}(n_{2j}+1)}\right)}$
$q_{2jk},\ j=\overline{1,m_2},k=\overline{1,n_{2j}}$	$\frac{1}{b_{2j}(n_{2j}+1))}\left(a_{2j}-p_{11}-n_{2j}v_{2jk}+ +\sum_{h=1,h\neq k}^{n_{2j}}v_{2jh}\right)$
$Q_{2j},\ j=\overline{1,m_2}$	$\frac{n_{2j}(a_{2j}-p_{11})-\sum_{k=1}^{n_{2j}}v_{2jk}}{b_{2j}(n_{2j}+1)}$

3.3. Nash equilibrium in the multilevel decentralized game

Let the decentralized tree-like supply chain be set with the certain number of levels. Analogous to the previous section the solution of the coordination problem we will begin with the analysis of the final junctures proceeding to the direction of the final peak.

Let us analyze the revenue function of the firm k from the juncture x_j^l:

$$\pi_{ljk} = q_{ljk}(p_{lj} - p_{it} - v_{ljk}), p_{it} : (l, j) \in S_t^i. \tag{21}$$

Let us substitute in the revenue formula (3.3.1) the formula for the variable p_{lj}, using the supply function (3):

$$\pi_{ljk} = q_{ljk}(a_{lj} - b_{lj}Q_{lj} - p_{it} - v_{ljk}). \tag{22}$$

Having done (21) (22) for all $k = \overline{1, n_{li}}$ and having applied the maximum condition of necessity:

$$\frac{\partial \pi_{ljk}}{\partial q_{ljk}} = 0, k = \overline{1, n_{lj}}, \tag{23}$$

we will result in the following system:

$$\begin{pmatrix} 2\;1\;1\cdots 1 \\ 1\;2\;1\cdots 1 \\ \vdots\;\vdots\;\vdots\;\;\vdots\;\;\vdots \\ 1\;1\;1\cdots 2 \end{pmatrix} \cdot \begin{pmatrix} q_{lj1} \\ q_{lj2} \\ \vdots \\ q_{ljn_{lj}} \end{pmatrix} = \begin{pmatrix} \frac{1}{b_{lj}}(a_{lj} - p_{it} - v_{lj1}) \\ \frac{1}{b_{lj}}(a_{lj} - p_{it} - v_{lj2}) \\ \vdots \\ \frac{1}{b_{lj}}(a_{lj} - p_{it} - v_{ljn_{1j}}) \end{pmatrix}. \tag{24}$$

System (24) has the matrix:

$$\begin{pmatrix} 2\;1\;1\cdots 1 \\ 1\;2\;1\cdots 1 \\ \vdots\;\vdots\;\vdots\;\;\vdots\;\;\vdots \\ 1\;1\;1\cdots 2 \end{pmatrix}_{[n_{lj} \times n_{lj}]}$$

that is non-degenerate due to the linear dependence of columns (rows). That is why system (24) can be solved in a one-valued way in correspondence to the variables $q_{ljk}, k = \overline{1, n_{lj}}$ and the unambiguous solution has the form:

$$\begin{pmatrix} q_{lj1} \\ q_{lj2} \\ \vdots \\ q_{ljn_{lj}} \end{pmatrix} = \begin{pmatrix} \frac{n_{lj}}{n_{lj}+1} & \frac{-1}{n_{lj}+1} & \cdots & \frac{-1}{n_{lj}+1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{n_{lj}+1} & \frac{-1}{n_{lj}+1} & \cdots & \frac{n_{lj}}{n_{lj}+1} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{b_{lj}}(a_{lj} - p_{it} - v_{lj1}) \\ \frac{1}{b_{lj}}(a_{lj} - p_{it} - v_{lj2}) \\ \vdots \\ \frac{1}{b_{lj}}(a_{lj} - p_{it} - v_{ljn_{lj}}) \end{pmatrix}$$

or after the multiplication of the solution has the form:

$$\begin{pmatrix} q_{lj1} \\ q_{lj2} \\ \vdots \\ q_{ljn_{lj}} \end{pmatrix} = \begin{pmatrix} \frac{1}{b_{lj}(n_{lj}+1)}\left(a_{lj} - \left(p_{it} + n_{lj}v_{lj1} - \sum_{h=2}^{n_{lj}} v_{ljh}\right)\right) \\ \frac{1}{b_{lj}(n_{lj}+1)}\left(a_{lj} - \left(p_{it} + n_{lj}v_{lj2} - \sum_{h=1,h\neq 2}^{n_{lj}} v_{ljh}\right)\right) \\ \vdots \\ \frac{1}{b_{lj}(n_{lj}+1)}\left(a_{lj} - \left(p_{it} + n_{lj}v_{ljn_{lj}} - \sum_{h=1}^{n_{lj}-1} v_{ljh}\right)\right) \end{pmatrix}. \tag{25}$$

For the juncture x_j^l the following par is valid as well

$$Q_{lj} = \sum_{k=1}^{n_{lj}} q_{ljk} = \frac{n_{lj}(a_{lj} - p_{it}) - \sum_{k=1}^{n_{lj}} v_{ljk}}{b_{lj}(n_{lj}+1)}. \tag{26}$$

Let us fulfill the same analogical operations (21) (26) for all the final peaks $x_j^l \in X_l$.

Now let us analyze the firm k from $x_j^{(l-1)}$. Its revenue function has the following form:

$$\pi_{(l-1)jk} = q_{(l-1)jk}\left(p_{(l-1)j} - p_{it} - v_{(l-1)jk}\right), k = \overline{1, n_{(l-1)j}}, \tag{27}$$

where $p_{it} : (l-1, j) \in S_t^i$.

Taking into consideration that the juncture $x_{(l-1)j}$ composes a sector, then from the condition of the deficit and surplus elimination (2) let us have the formula

$$\begin{aligned} \sum_{k=1}^{n_{(l-1)j}} q_{(l-1)jk} = Q_{(l-1)j} = \sum_{h:(l,h)\in S_j^{l-1}} Q_{lh} = \\ = \sum_{h:(l,h)\in S_j^{l-1}} \frac{n_{lh}\left(a_{lh} - p_{(l-1)j}\right) - \sum_{r=1}^{n_{lh}} v_{lhr}}{b_{lh}(n_{lh}+1)}, \end{aligned} \tag{28}$$

from that it is possible to express the variable $p_{(l-1)j}$ in one-valued terms:

$$\begin{aligned} p_{(l-1)j} = f_{(l-1)j}\left(q_{(l-1)j1}, \ldots, q_{(l-1)jn_{(l-1)j}}\right) = \\ = \frac{-\sum_{k=1}^{n_{(l-1)j}} q_{(l-1)jk} + \sum_{h:(l,h)\in S_j^{l-1}} \frac{n_{lh}a_{lh} - \sum_{r=1}^{n_{lh}} v_{lhr}}{b_{lh}(n_{lh}+1)}}{\sum_{h:(l,h)\in S_j^{l-1}} \frac{n_{lh}}{b_{lh}(n_{lh}+1)}}. \end{aligned} \tag{29}$$

Let us substitute (29) in the revenue formulas (27)

$$\pi_{(l-1)jk} = q_{(l-1)jk}\left(f_{(l-1)j} - p_{it} - v_{(l-1)jk}\right), k = \overline{1, n_{(l-1)j}}, \tag{30}$$

and let us apply the maximum condition of necessity to the formulas (30):

$$\frac{\partial \pi_{(l-1)jk}}{\partial q_{(l-1)jk}} = f_{(l-1)j} - p_{it} - v_{(l-1)jk} + \\ + q_{(l-1)jk} \cdot \frac{-1}{\sum\limits_{h:(l,h)\in S_j}^{l-1} \frac{n_{lh}}{b_{lh}(n_{lh}+1)}} = 0,\ k = \overline{1, n_{(l-1)j}}, \tag{31}$$

or in the matrix form:

$$\begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \vdots & 2 \end{pmatrix} \cdot \begin{pmatrix} q_{(l-1)j1} \\ q_{(l-1)j2} \\ \vdots \\ q_{((l-1)jn_{(l-1)j}} \end{pmatrix} =$$

$$= \begin{pmatrix} \sum\limits_{h:(l,h)\in S_j}^{l-1} \frac{1}{b_{lh}(n_{lh}+1)} \left(n_{lh}a_{lh} - n_{lh}p_{it} - n_{lh}v_{(l-1)j1} - \sum\limits_{r=1}^{n_{lh}} v_{lhr} \right) \\ \sum\limits_{h:(l,h)\in S_j}^{l-1} \frac{1}{b_{lh}(n_{lh}+1)} \left(n_{lh}a_{lh} - n_{lh}p_{it} - n_{lh}v_{(l-1)j2} - \sum\limits_{r=1}^{n_{lh}} v_{lhr} \right) \\ \vdots \\ \sum\limits_{h:(l,h)\in S_j}^{l-1} \frac{1}{b_{lh}(n_{lh}+1)} \left(n_{lh}a_{lh} - n_{lh}p_{it} - n_{lh}v_{(l-1)jn_{(l-1)j}} - \sum\limits_{r=1}^{n_{lh}} v_{lhr} \right) \end{pmatrix}. \tag{32}$$

Since the matrix of the system (32)

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}_{\left[n_{(l-1)i} \times n_{(l-1)i}\right]}$$

is a non-degenerate one due to having linear independence of the columns (rows) the opposite matrix exists:

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}^{-1}_{\left[n_{(l-1)j} \times n_{(l-1)j}\right]} =$$

$$= \begin{pmatrix} \frac{n_{(l-1)j}}{n_{(l-1)j}+1} & \frac{-1}{n_{(l-1)j}+1} & \cdots & \frac{-1}{n_{(l-1)j}+1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{n_{(l-1)j}+1} & \frac{-1}{n_{(l-1)j}+1} & \cdots & \frac{n_{(l-1)j}}{n_{(l-1)j}+1} \end{pmatrix}_{n_{(l-1)j} \times n_{(l-1)j}} \quad (33)$$

As a result of that, (3.3.10) could be solved in a one-valued way in relation to the variables $q_{(l-1)jk}, k = \overline{1, n_{(l-1)j}}$:

$$q_{(l-1)jk} = \frac{1}{n_{(l-1)j}+1} \left[\sum_{h:(l,h)\in S_j}^{l-1} \frac{1}{b_{lh}(n_{lh}+1)} \left(n_{lh}a_{lh} - n_{lh}p_{it} - \right. \right.$$
$$\left.\left. - \sum_{r=1}^{n_{lh}} v_{lhr} - n_{(l-1)j}n_{lh}v_{(l-1)jk} + n_{lh} \sum_{e=1, e\neq k}^{n_{(l-1)j}} v_{(l-1)je} \right) \right], \; k = \overline{1, n_{(l-1)j}}. \quad (34)$$

There are could be further calculated the value of $Q_{(l-1)j}$:

$$Q_{(l-1)j} = \sum_{k=1}^{n_{(l-1)j}} q_{(l-1)jk} = \frac{1}{n_{(l-1)j}+1} \left[\sum_{h:(l,h)\in S_j^{l-1}} \frac{1}{b_{lh}(n_{lh}+1)} \times \right.$$
$$\left. \times \left(n_{(l-1)j} \left(n_{lh}a_{lh} - n_{lh}p_{it} - \sum_{r=1}^{n_{lh}} v_{lhr} \right) - n_{lh} \sum_{k=1}^{n_{(l-1)j}} v_{(l-1)jk} \right) \right]. \quad (35)$$

Let us repeat the process (27) (35) for all the remained junctures x_i^{l-1} from the same level: $x_i^{l-1} \in X_{l-1}, i \neq j$.

Then we by the similar way will analyze the peaks x_t^i from multitudes X_i peaks of the level i, $i = (l-2), (l-3), \ldots, 2$, will solve the two level subgame in each of the sectors that we created by these junctures, having received the solution depending on the supplier price of the juncture x_t^i and express the meaning of this price in terms of the variables from the volume juncture.

Let us proceed to the analysis of the multitude in the first level peaks $X_1 = \{x_1^1\}$. The revenue functions view for the certain firm k from the juncture x_1^1 has the view:

$$\pi_{11k} = q_{11k}(p_{11} - v_{11k}). \quad (36)$$

Let us consider that the variable p_{11} has the expression by the variables $q_{11k}, k = \overline{1, n_{11}}$ and the parameters of the production costs that can be received after the consideration of all X_i, $i = \overline{2, l-1}$ from the condition of the deficit and surplus nonexistence:

$$p_{11} = f_{11}(q_{111}, \ldots, q_{11n_{11}}, v_{it1}, \ldots, v_{itn_{it}}, \ldots, v_{111}, \ldots, v_{11n_{11}}),\quad i, t : (i, t) \in S_1^1, \tag{37}$$

where f_{11} is the linear function depending on the arguments $q_{111}, \ldots, q_{11n_{11}}$.

Let us substitute the equation (37) in the revenue function (36)

$$\pi_{11j} = q_{11k}\left(f_{11}(q_{111}, \ldots, q_{11n_{11}}, v_{it1}, \ldots, v_{itn_{it}}, \ldots, v_{11n_{11}}) - v_{1k}\right), \tag{38}$$

and apply to the (38) the maximum condition of necessity:

$$\begin{aligned}\frac{\partial \pi_{11k}}{\partial q_{11k}} &= f_{11}(q_{111}, \ldots, q_{11n_{11}}, v_{it1}, \ldots, v_{itn_{it}}, \ldots, v_{111}, \ldots, v_{11n_{11}}) - \\ &- v_{11k} + q_{11k}\frac{\partial f_{11}}{\partial q11k} = 0, k = \overline{1, n_{11}},\end{aligned} \tag{39}$$

As this takes place the meanings of all derivatives $\frac{\partial f_{11}}{\partial q_{11k}}$, $k = \overline{1, n_{11}}$ are constant due to the linearity of the function f_{11}. The system (3.3.15) is the linear equations system relative to $q_{111}, \ldots, q_{11n_{11}}$ with a nondegenerate matrix

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}_{[n_{11} \times n_{11}]} \tag{40}$$

and due to that it is uniquely solvable in relation to all $q_{11k}, k = \overline{1, n_{11}}$ where this solution depends only on the predefined supply chain parameters. Then by consequently substituting the deduced meanings to the equations for the unknown variables we will find their equilibrium meanings. Hence, the optimal flow d^* is found and the task of coordination to the decentralized model of the multilevel supply chains is solved.

4. Coordination of the centralized multilevel supply chain

Let the certain multilevel supply chain with the tree-like distributive structure be defined. Let us assume that all its participants are joining the coalition and deciding to act in coordination having the goal of the total profit functions maximization in the overall supply chain under the known linear supply functions in the finite junctures.

For each of the firms from this chain let us write down its revenue function $\pi_{ijk}(d)$, and then let us sum them by $i = \overline{1, l}$, $j = \overline{1, m_i}$, $k = \overline{1, n_{ij}}$ in order to find the overall supply chain revenue $\Pi(d)$. Then it is necessary to find that feasible flow $\widehat{d}$ that can contribute to the satisfaction of the formula

$$argmax_{d \in D} \Pi(d) = \widehat{d},$$

leading us to the optimization problem under the following conditions:

$$\max_{d\in D} \Pi(d) = \max_{q_{ijh},p_{ij}} \left(\sum_{i=2}^{l} \sum_{j=1}^{m_i} \sum_{k=1}^{n_{ij}} \pi_{ijk} \left(q_{ij1}, \ldots, q_{ijn_{ij}}, v_{ij1}, \ldots, v_{ijn_{ij}}, p_{ij}, p_{th}\right) + \right.$$
$$\left. + \sum_{k=1}^{n_{11}} \pi_{11k} \left(q_{111}, \ldots, q_{11n_{11}}, v_{111}, \ldots, v_{11n_{11}}, p_{11}\right) \right), \ p_{th} : (i,j) \in S_h^t; \tag{41}$$

$$p_{lj} = a_{lj} - b_{lj} \sum_{k=1}^{n_{lj}} q_{ljk}, j = \overline{1, m_l}; \tag{42}$$

$$\sum_{r=1}^{n_{th}} q_{thr} = \sum_{i,j:(i,j)\in S_h^t} \sum_{k=1}^{n_{ij}} q_{ijk}, t, h : x_h^t \notin X_l; \tag{43}$$

$$q_{ijk} \geq 0, i = \overline{1,l}, j = \overline{1, m_i}, k = \overline{1, n_{ij}}; \tag{44}$$

$$p_{ij} \geq 0, i = \overline{1,l}, j = \overline{1, m_i}. \tag{45}$$

From the properties of the maximizing function $\Pi(d)$ and view of the constraints (42) (45) we conclude that (41) (45) is the linear optimization problem under the linear constraints of equation and inequation types.

For the solution of the analyzed optimization problem there was a program created in the MATLAB environment. This program realized the interactive search algorithm of the maximum point search under the constraints of equation and inequation types based on the sequential quadratic programming method.

Optimization problem (41) (45) (and, as a consequence, results received after its solution) has only one, but very substantial, drawback: it requires after the usage an additional imputation system, because under the received optimal volumes that are really minimizing the revenue on the whole supply chain, the revenue of the certain participants is pertaining to zero or negative. That is why after the optimal flow to the chain identification it is necessary to imply the contract system among all the participants which states explicitly the imputation of the total revenue received. However, it is very often difficult to implement that in real life.

Let us analyze the method using an alternative definition of the optimization problem and not requiring after it usage of any mathematical instruments.

5. Formalization of coordination attitude with the weighted Nash solution usage

Let us have the game in the standard form, namely the plurality $\Gamma = \langle N, \{Y_i\}_{i\in N}, \{H_i\}_{i\in N}\rangle$, where $N = \{1, 2, \ldots, n\}$ is a nonvacuous set of players, Y_i is the set of players i strategies, and H_i is a payoff functional of the player i that is defined on the Cartesian product of sets $\{Y_i\}_{i\in N}$ for the strategies of players $Y = \prod_{i\in N} Y_i$, $H_i : Y \to R$. Simply ordered plurality $N = \{x_1^1, x_1^2, x_2^2, \ldots, x_{m_l}^l\}$ for all the junctures of the supply chain we will consider as the plurality of players and pluralities U_{ij}, defined by formula (4) pluralities for strategies of players $x_j^i \in N$. Let us for each player $x_j^i \in N$ define in accordance the vector $\pi_j^i = \left(\pi_{ij1}, \pi_{ij2}, \ldots, \pi_{ijn_{ij}}\right)$

and in terms of players payoff functional let us take the mix of these vectors, simply ordered according to the ordering of the players plurality $\pi = \{\pi_1^1, \dots, \pi_{m_l}^l\}$.

Let us call $\pi^* = \left\{\pi_{ijk}^*\right\}_{i,j,k}$ as the revenue of all the supply chain participants that is gained in decentralized solution of a coordination problem in the same supply chain. Let us create the function

$$\Phi(d) = \prod_{i=1}^{l}\prod_{j=1}^{m_i}\prod_{k=1}^{n_{ij}} \left(\pi_{ijk}(d) - \pi_{ijk}^*\right)^{\alpha_{ijk}},$$

where α_{ijk} are certain numbers such as $\alpha_{ijk} > 0, \forall i = \overline{1,l}, j = \overline{1,m_i}, k = \overline{1,n_{ij}}$ and

$$\sum_{i=1}^{l}\sum_{j=1}^{m_i}\sum_{k=1}^{n_{ij}} \alpha_{ijk} = 1.$$

Then the solution of the following optimization problem with constraints is, on the one hand, the weighted Nash solution and on the other is the Pareto-optimal flow in the supply chain:

$$\begin{aligned} \max_{q_{ijh},p_{ij}} & \left[\left(\prod_{i=2}^{l}\prod_{j=1}^{m_i}\prod_{k=1}^{n_{ij}} \left(\pi_{ijk}\left(q_{ij1},\dots,q_{ijn_{ij}},v_{ij1},\dots,v_{ijn_{ij}},p_{ij},p_{th}\right) - \pi_{ijk}^*\right)^{\pi_{ijk}}\right) \times \right. \\ & \left. \times \left(\prod_{k=1}^{n_{11}} \left(\pi_{11k}\left(q_{111},\dots,q_{11n_{11}},v_{111},\dots,v_{11n_{11}},p_{11}\right) - \pi_{11k}^*\right)^{\pi_{11k}}\right)\right], \\ & p_{th} : (i,j) \in S_h^t; \end{aligned} \tag{46}$$

$$\pi_{ijk} \geq \pi_{ijk}^*, \; i = \overline{1,l}, \; j = \overline{1,m_i}, k = \overline{1,n_{ij}}; \tag{47}$$

$$p_{lj} = a_{lj} - b_{lj}\sum_{k=1}^{n_{lj}} q_{ljk}, \; j = \overline{1,m_l}; \tag{48}$$

$$\sum_{r=1}^{n_{th}} q_{thr} = \sum_{i,j:(i,j)\in S_h^t}\sum_{k=1}^{n_{ij}} q_{ijk}, \; t,h : x_h^t \in X_l; \tag{49}$$

$$q_{ijk} \geq 0, \; i = \overline{1,l}, \; j = \overline{1,m_i}, \; k = \overline{1,n_{ij}}; \tag{50}$$

$$p_{lj} \geq 0, \; j = \overline{1,m_i} \tag{51}$$

For the solution of this linear optimization problem with the non-linear constraints there was a program created in MATLAB that is representing the iterative search of optimal solution with the predefined constraints in the kind of equations and inequations with the usage of the sequential quadratic programming method as the most effective method of the linear functions constrained optimization.

6. Example and comparison of the solutions

Let us look at the specific example of the supply chain and compare the solutions that were received after each of the methods were implemented. Let us have the supply chain depicted on the Table 2.

Table 2. Meanings of the supply chain parameters

	Juncture x_1^1	Juncture x_1^2	Juncture x_1^3	Juncture x_2^3
Number of firms in the juncture, n_{ij}	$n_{11} = 2$	$n_{21} = 1$	$n_{31} = 4$	$n_{32} = 2$
Meaning of costs to the single unit of good production, v_{ijh}	$v_{111} = 1500$ $v_{112} = 1505$	$v_{211} = 700$	$v_{311} = 342$ $v_{312} = 340$ $v_{313} = 338$ $v_{314} = 345$	$v_{321} = 120$ $v_{322} = 122$

Let us find consequent decentralized solution for this supply chain, then centralized, and finally Nash solution in which as the weight coefficients there will be the following numbers used:

$$\alpha_{111} = \alpha_{112} = \frac{1}{3};$$

$$\alpha_{111} = \alpha_{112} = \frac{1}{3};$$

$$\alpha_{211} = \frac{2}{9};$$

$$\alpha_{311} = \alpha_{312} = \alpha_{313} = \alpha_{314} = \alpha_{321} = \alpha_{322} = \frac{1}{54}.$$

Comment. These numbers were received by the authors algorithm of the number crunching in the weighted coefficients, according to which the largest weight is assigned to the root juncture, and then the weights are decreasing by the movement from the level to level.

Let us find the decentralized solution for this example. Revenue function for all the firms from the juncture of the 3rd level have the type (52) and (53):

$$\pi_{311} = q_{311}\left(5000 - 0,25\sum_{j=1}^{4} q_{31j} - p_{11} - 342\right), \tag{52}$$

$$\pi_{312} = q_{312}\left(5000 - 0,25\sum_{j=1}^{4} q_{31j} - p_{11} - 340\right),$$

$$\pi_{313} = q_{313}\left(5000 - 0,25\sum_{j=1}^{4} q_{31j} - p_{11} - 338\right),$$

$$\pi_{314} = q_{314}\left(5000 - 0,25\sum_{j=1}^{4} q_{31j} - p_{11} - 345\right);$$

$$\pi_{321} = q_{321}\left(6000 - 0,09\sum_{j=1}^{2} q_{32j} - p_{21} - 120\right), \quad (53)$$

$$\pi_{322} = q_{322}\left(6000 - 0,09\sum_{j=1}^{2} q_{32j} - p_{21} - 122\right).$$

Let us apply to all the functions in (6.1) and (6.2) the maximum condition of necessity and deduce the two sets of equations respectively:

$$\begin{pmatrix} 0,5 & 0,25 & 0,25 & 0,25 \\ 0,25 & 0,5 & 0,25 & 0,25 \\ 0,25 & 0,25 & 0,5 & 0,25 \\ 0,25 & 0,25 & 0,25 & 0,5 \end{pmatrix} = \begin{pmatrix} q_{311} \\ q_{312} \\ q_{313} \\ q_{314} \end{pmatrix} = \begin{pmatrix} 4658 - p_{11} \\ 4660 - p_{11} \\ 4662 - p_{11} \\ 4655 - p_{11} \end{pmatrix}. \quad (54)$$

$$\begin{pmatrix} 0,18 & 0,09 \\ 0,09 & 0,18 \end{pmatrix} \begin{pmatrix} q_{321} \\ q_{322} \end{pmatrix} = \begin{pmatrix} 5880 - p_{21} \\ 5878 - p_{21} \end{pmatrix}; \quad (55)$$

After solving the systems (54) and (55) we have the formula for q_{3ij}:

$$\begin{cases} q_{311} = 3724 - 0,8p_{11}, \\ q_{312} = 3732 - 0,8p_{11}, \\ q_{313} = 3740 - 0,8p_{11}, \\ q_{31}4 = 3712 - 0,8p_{11}. \end{cases} \quad (56)$$

$$\begin{cases} q_{321} = \frac{1}{27}(588200 - 100p_{21}), \\ q_{322} = \frac{1}{27}(587600 - 100p_{21}). \end{cases} \quad (57)$$

Because of the deficit and surplus mitigation condition we will receive the formula

$$Q_{32} = q_{321} + q_{322} = \frac{1175800}{27} - \frac{200}{27}p_{21} = Q_{21} = q_{211},$$

from that one can express meaning of the variable p_{21}

$$p_{21} = 5879 - 0,135q_{211}. \quad (58)$$

For the unique firm out of the juncture x_1^2 revenue function is written in the form of the formula:

$$\pi_{211} = q_{211}\left(p_{21} - p_{11} - 700\right),$$

substituting in which the equation (58), we will find:

$$\pi_{211} = q_{211}\left(5879 - 0,135q_{211} - p_{11} - 700\right).$$

Implementation of the maximum condition of necessity to this equation will be resulted in the par:

$$\frac{\partial \pi_{211}}{\partial q_{211}} = 0 \Longrightarrow q_{211} = \frac{517900}{27} - \frac{100}{27}p_{11}. \quad (59)$$

The condition of the surplus and deficit mitigation in the root peak center can be written in the form of equation

$$Q_{11} = q_{111} + q_{112} = Q_{21} + Q_{31} = q_{211} + \sum_{i=1}^{4} q_{31i},$$

from that after having substituted (6.5) in (6.8) one can express p_{11}:

$$p_{11} = \frac{1150520}{233} - \frac{135}{932}\left(q_{111} + q_{112}\right) \tag{60}$$

Firms 1 and 2 from the root sector have the following revenue functions respectively:

$$\pi_{111} = q_{111}\left(p_{11} - 1500\right), \tag{61}$$

$$\pi_{112} = q_{112}\left(p_{11} - 1505\right), \tag{62}$$

which after the plugging in (60) will have the form:

$$\pi_{111} = q_{111}\left(\frac{1150520}{233} - \frac{135}{932}\left(q_{111} + q_{112}\right) - 1500\right), \tag{63}$$

$$\pi_{112} = q_{112}\left(\frac{(1150520}{233} - \frac{135}{932}\left(q_{111} + q_{112}\right) - 1505\right). \tag{64}$$

After the implementation of the maximum condition of necessity to the (63) and (64) we will receive a system:

$$\begin{pmatrix} \frac{135}{932} & \frac{135}{466} \\ \frac{135}{466} & \frac{135}{932} \end{pmatrix} \begin{pmatrix} q_{111} \\ q_{112} \end{pmatrix} = \begin{pmatrix} \frac{801020}{233} \\ \frac{799855}{233} \end{pmatrix},$$

the unique solution of which has the form:

$$\begin{cases} q_{111} = \frac{213916}{27} \approx 7923, \\ q_{112} = \frac{212984}{27} \approx 7889. \end{cases} \tag{65}$$

Let us substitute the found meanings (65) in the equations (60)

$$p_{11} = \frac{616895}{233} \approx 2648. \tag{66}$$

Let us substitute the meaning (66) in the (56) and (59) so that we will come up with the following:

$$\begin{aligned} q_{311} &= \frac{374176}{233} \approx 1606, \\ q_{312} &= \frac{376040}{233} \approx 1614, \\ q_{313} &= \frac{377904}{233} \approx 1622, \\ q_{314} &= \frac{371380}{233} \approx 1594, \end{aligned} \tag{67}$$

$$q_{211} = \frac{19660400}{2097} \approx 9375; \tag{68}$$

Then, after having substituted (68) in the formula (58) we will find the meaning for $p_2 1$:

$$p_{21} = \frac{1074901}{233} \approx 4613. \tag{69}$$

By substituting (68) to the formula (58) let us find q_{321} and q_{322} from the equation (57):

$$q_{321} = \frac{3284500}{699} \approx 4699; \tag{70}$$

$$q_{322} = \frac{9806900}{2097} \approx 4677.$$

Finally, using the calculated meanings of production volumes (67) and (70) in the finite junctures x_1^3 and x_2^3, let us calculate the meaning of optimal prices p_{31} and p_{32} with the usage of the supply function:

$$p_{31} = \frac{790125}{233} \approx 3391,$$

$$p_{32} = \frac{1201396}{233} \approx 5156.$$

Knowing the equilibrium meanings of all variables, we can calculate the revenue of every participant and then receive the overall supply chain revenue that is equal to:

$$\Pi^d = \frac{53525765475416}{1465803} \approx 3.6516 \cdot 10^7. \tag{71}$$

Now let us find the optimal meanings by solving with the help of MATLAB platform the total revenue maximization problem in case of decentralized supply chain model and the maximization of the weighted Nash solution problem. Let us place all the received meanings in the single table (Table 3) for the intuitive comparison.

While comparing the meanings of the total chain revenue in decentralized and centralized models, let us notice that in case of centralized participants behavior the total revenue of chain has been increased to $1,1042 \cdot 107$ or approximately to 30%. The number of analogous numerative experiments has found out that the chain centralization has on average the 25% gain in terms of total revenue in comparison with decentralized model. What is more, it is clear from the table that the Nash weighted arbitrage solution has increased the total profit of supply chain approximately to 29% from the revenue meaning in the Nash equilibrium in decentralized model. This result is a bit worse that has been received by the means of overall supply chain revenue maximization problem solution. However, the Nash weighted arbitrage solution guarantees for each of participants the positive gain and does not require an imputation procedure.

Table 3. Meanings of variables and revenue

	Nash equilibrium	Solution of the total profit maximization problem	Nash weighted arbitrage solution
Juncture x_1^1			
Volume of output	$q_{111} \approx 7923$ $q_{112} \approx 7888$	$q_{111} \approx 27566$ $q_{112} \approx 0$	$q_{111} \approx 13629$ $q_{112} \approx 13126$
Price	$p_{11} \approx 2648$	$p_{11} \approx 1553$	$p_{11} \approx 2402$
Revenue of participants	$\pi_{111} \approx 9092365$ $\pi_{112} \approx 9013310$	$\pi_{111} \approx 60461350$ $\pi_{112} \approx 0$	$\pi_{111} \approx 12299768$ $\pi_{112} \approx 11780161$
Juncture x_1^2			
Volume of output	$q_{211} \approx 9375$	$q_{211} \approx 21242$	$q_{211} \approx 21441$
Price	$p_{21} \approx 4613$	$p_{21} \approx 0$	$p_{21} \approx 3716$
Revenue of participants	$\pi_{211} \approx 118664742$	$\pi_{211} \approx 16198593$	$\pi_{211} \approx 13144816$
Juncture x_1^3			
Volume of output	$q_{311} \approx 1606,$ $q_{312} \approx 1614,$ $q_{313} \approx 1622,$ $q_{314} \approx 1594$	$q_{311} \approx 0,$ $q_{312} \approx 0,$ $q_{313} \approx 6324,$ $q_{314} \approx 0$	$q_{311} \approx 696$ $q_{312} \approx 1193$ $q_{313} \approx 2424$ $q_{314} \approx 1001$
Price	$p_{31} \approx 3391$	$p_{31} \approx 3419$	$p_{31} \approx 3671$
Revenue of participants	$\pi_{311} \approx 644733$ $\pi_{312} \approx 651173$ $\pi_{313} \approx 657644$ $\pi_{314} \approx 635134$	$\pi_{311} \approx 0$ $\pi_{312} \approx 0$ $\pi_{313} \approx -4285648$ $\pi_{314} \approx 0$	$\pi_{311} \approx 644770$ $\pi_{312} \approx 1108408$ $\pi_{313} \approx 2256850$ $\pi_{314} \approx 925272$
Juncture x_2^3			
Volume of output	$q_{321} \approx 4699,$ $q_{322} \approx 4677$	$q_{321} \approx 21242,$ $q_{322} \approx 0$	$q_{321} \approx 12018$ $q_{322} \approx 9423$
Price	$p_{32} \approx 5156$	$p_{32} \approx 4088$	$p_{32} \approx 4070$
Revenue of participants	$\pi_{321} \approx 1987132$ $\pi_{322} \approx 1968381$	$\pi_{321} \approx -24758272$ $\pi_{322} \approx 0$	$\pi_{321} \approx 2821735$ $\pi_{322} \approx 2193425$
Total chain: revenue	$\approx 3,65 \cdot 10^7$	$\approx 4,76 \cdot 10^7$	$\approx 4,72 \cdot 10^7$

7. Conclusions

Within this paper we have analyzed supply chains with the tree-like distributive structure, where each juncture of this chain represents the competitive firms plurality that are producing and consuming the homogeneous product and that are having different production costs, but at the same time junctures do not compete with each other. It was assumed that the markets where the final products are realized by the finite junctures, do not compete with each other and function under the Cournot model with linear supply functions. We have discussed the question of participants coordination, i.e. the of the problem concerning the choice of such strategies that are satisfying the predefined optimality criteria. The mathematical formalization of the multilevel tree-like supply chains with the help of tree-like graph was conducted and the three solutions to the coordination problem were proposed: decentralized solution, centralized solution and weighted Nash solution. The search for decentralized solution has resulted in absolute Nash equilibrium being found in the multilevel hierarchical fully equipped with information game for which we have created the algorithm of this equilibrium solution finding. For the case of the centralized participants behavior in the supply chain with the analyzed structure, the coordination problem was formulated as the problem of non-linear conditional optimization. Numerical simulation has found that such an approach increases the total revenue of the supply chain on average at 25%, but is does not guarantee the positive gain to all of the participants, so requires the imputation system to be implemented. The analysis of results having received from the numerical simulation, has forced us to find an alternative approach to the supply chain coordination. Acting as such an approach the Nash weighted solution was chosen that, as it was found out experimentally, even though gives a smaller gain in terms of revenue than the one examined earlier, but guarantees the positive gain to all of the participants.

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