Contributions to Game Theory and Management, X, 339–349

Constructive and Blocking Powers in Some Applications

Svetlana Tarashnina[1] **and Nadezhda Smirnova**[2]

[1] *St.Petersburg State University,*
7/9 Universitetskaya nab., St.Petersburg, 199034, Russia
E-mail: tarashnina@gmail.com
[2] *Higher School of Economics*
Department of Applied Mathematics and Business Informatics,
Soyuza Pechatnikov ul. 16, St. Petersburg, 190008, Russia
E-mail: nadezhda.v.smirnova@gmail.com

Abstract We investigate the prenucleolus, the anti-prenucleolus and the SM-nucleolus in glove market games and weighted majority games. This kind of games looks desirable for considering solution concepts taking into account the blocking power of a coalition S with different weights. Analytical formulae for calculating the solutions are presented for glove market game. Influence of the blocking power on players' payoffs is discussed and the examples which demonstrate similarities and differences comparing with other solution concepts are given.

Keywords: cooperative TU-game, solution concept, prenucleolus, SM-nucleolus, constructive and blocking power, glove market game, weighted majority game.

1. Introduction

In this paper we consider two types of classical cooperative games "glove market game" and "weighted majority game". Our purpose is to look for the ways to distribute the value $v(N)$ according to some excess-based solution concepts of TU-games such as the prenucleolus (Schmeidler, 1969), the anti-prenucleolus and the SM-nucleolus (Tarashnina, 2011). The interest to these solution concepts is associated with different taking into account the constructive power and the blocking power of a coalition $S \subseteq N$ in the game. In fact,

- the prenucleolus takes into account only the constructive power;
- the anti-prenucleolus takes into account only the blocking power;
- the SM-nucleolus takes into account the average of the constructive and the blocking power.

Comparing the allocations obtained by the mentioned concepts, we attempt to answer what do they represent in the considered games, their similarities and differences. In particular, we evaluate the impact of the constructive and the blocking power on payoffs of players.

Unfortunately, there exists no analytic formulae for the excess-based solutions which have to be computed numerically by solving a series of linear programming problems. The existence of analytic formulae for calculating solutions allow to find allocations directly and analyze them. Regarding to glove market games we use formulae from Tarashnina and Sharlai (2015). For weighted majority games we apply the algorithm from Britvin and Tarashnina (2013).

The paper is structured as follows. Section 1 contains notions and definitions of cooperative game theory. Section 2 is devoted to glove market games. Apart from analytic formulae some illustrative examples are presented and discussed there. In Section 3 we switch reader's attention to weighted majority games and analyze the considered solutions in this class of games. Finally, we give some insight into the blocking power of a "weak" and "strong" players in TU-games.

2. Cooperative game theory concepts

In this paper we deal with cooperative games with transferable utility, or simply TU-games. A cooperative TU-game is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of players and $v : 2^N \to R^1$ is a characteristic function with $v(\emptyset) = 0$. Here $2^N = \{S \subseteq N\}$ is the set of coalitions in (N, v). Since the game (N, v) is completely determined by the characteristic function v, we shall sometimes represent a TU-game by its characteristic function v. Let G^N be the set of TU-games with a finite set of players N.

Due to the classical cooperative approach we look for the ways to distribute the amount $v(N)$ over the members of the grand coalition. A corresponding vector of payoffs (or a set of vectors) that distributes the amount $v(N)$ among the players is called a solution of the game. Here we consider solutions that belong to the set $X^0(N, v)$ of preimputations of a game (N, v), i.e. $X^0(N, v) = \{x \in R^n : \ x(N) = v(N)\}$.

Let x be a preimputation in a game (N, v). The excess $e(x, v, S)$ of a coalition S at x is $e(x, v, S) = v(S) - x(S)$. Due to Maschler (1992), the excess of a coalition evaluates a measure of dissatisfaction of the coalition at preimputation x, which should be minimized. For each $z \in R^n$ we define the vector $\theta(z) \in R^n$, which arises from z by arranging its components in a non-increasing order.

Definition 1. The prenucleolus of a game (N, v) is the set of vectors in $X^0(v)$ whose $\theta(e(x, v, S)_{S\subseteq N})$'s are lexicographically least, i.e.

$$\mathcal{N}(v) = \{x \in X^0(v) : \ \theta\big(e(x, v, S)_{S\subseteq N}\big) \preceq_{lex} \theta\big(e(y, v, S)_{S\subseteq N}\big) \text{ for all } y \in X^0(v)\}.$$

The prenucleolus of a game is a singleton (Schmeidler, 1969), so we denote this single point by $\nu(v)$. From Definition 1 it follows that the prenucleolus doesn't take into account the blocking power of a coalition. This allocation method is based on a notion of constructive power. The meaning of the constructive power relates to the amount $v(S)$ and it is the worth of coalition S, or to be exact what S can reach by cooperation.

Two allocation methods that consider the blocking power are the SM–nucleolus and the anti-prenucleolus. By the blocking power of coalition S we understand the difference between $v(N)$ and $v(N \setminus S)$ — the amount $v^*(S)$ that the coalition S brings to N if the last is formed — its contribution to the grand coalition.

Given a cooperative TU-game (N, v), the dual game (N, v^*) of (N, v) is defined by

$$v^*(S) = v(N) - v(N \setminus S)$$

for all coalitions $S \subseteq N$. Then, the dual excess $e(x, v^*, S)$ of a coalition $S \subseteq N$ at x is $e(x, v^*, S) = v^*(S) - x(S)$ where x is a preimputation in (N, v).

Definition 2. The anti-prenucleolus of a game (N, v) is defined as

$$\psi(N, v) = \{x \in X^0(N, v) : \theta(e(x, v^*, S) \prec_{lex} \theta(e(y, v^*, S) \text{ for all } y \in X^0(N, v)\},$$

where $\theta(e(x, v^*, S)_{S \subseteq N})$ is a vector of excesses which components are arranged in non-increasing order.

The anti-prenucleolus takes into account only the blocking power of each coalition. Clearly, the anti-prenucleolus of a game (N, v) can be defined as the prenucleolus of the dual game (N, v^*).

In order to define the SM-nucleolus, we consider the weighted sum-excess of a coalition $S \subseteq N$ at each $x \in X^0(N, v)$ as follows

$$\overline{e}(x, v, S) = \frac{1}{2}e(x, v, S) + \frac{1}{2}e(x, v^*, S).$$

Definition 3. The SM-nucleolus of a game (N, v) is defined as

$$\mu(N, v) = \{x \in X^0(N, v) : \theta(\overline{e}(x, v, S) \prec_{lex} \theta(\overline{e}(y, v, S) \text{ for all } y \in X^0(N, v)\},$$

where $\theta(\overline{e}(x, v, S)_{S \subseteq N})$ is a vector of sum-excesses which components are arranged in non-increasing order.

Here the weights for the constructive and the blocking power are equal to $\frac{1}{2}$. However, these weights can be arbitrary, what has been shown in Smirnova and Tarashnina (2012), Smirnova and Tarashnina (2016).

Notice that the SM-nucleolus coincides with the prenucleolus of the constant-sum game (N, w) where $w = \dfrac{v + v^*}{2}$ (Tarashnina, 2011).

3. Glove market game

A glove game is one of the most popular market games in cooperative game theory that was proposed by Shapley and Shubik (1969). This game describes the following situation: there are two types of complementary products on the market and a finite set of firms each of which can produce a product of one type. A customer needs both types of products.

Let N consists of two types of players $N = P \cup Q$ where P and Q are disjunct sets of players. Each player of P owns a right-hand glove and each player of Q owns a left-hand glove. If j members of P and k members of Q form a coalition, they have $\min\{j, k\}$ complete pairs of gloves, each being worth 1. Unmatched gloves are worth nothing.

The characteristic function for the game is defined by

$$v(S) = \min\{|P \cup S|, |Q \cup S|\}, \quad S \subset N\,. \tag{1}$$

Thus, the worth of a coalition S is equal to the number of pairs of gloves the coalition can assemble. Without loss of generality, assume $|P| \geq |Q|$.

This model is quite popular in cooperative game theory and its core as well as other classical solution concepts have been already studied (see Owen, 1975; Aumann and Shapley, 1974; Billera and Raanan, 1981; Einy et al., 1996).

The core represents a payoff vector where the holders of the scarce commodity (the left-glove owners in our case) obtain a payoff of 1 and the other players obtain nothing. This result holds for $|P| = 100$ and $|Q| = 99$ as well as for $|P| = 100$ and $|Q| = 1$. The same result is fulfilled for the prenucleolus since it belongs to the core (if the last is nonempty). The following result holds.

Theorem 1. *Let (N, v) be a glove market game with characteristic function (1). Suppose that $|P| = p$, $|Q| = q$, $P = \{i_1, \ldots, i_p\}$, $Q = \{j_1, \ldots, j_q\}$, $p > q$. Then, the prenucleolus $\nu(v)$ is defined by the formulae*

$$\text{for } \; p > q \quad \nu_{i_k} = 0, \quad \nu_{j_l} = 1, \tag{2}$$

$$\text{for } \; p = q \quad \nu_{i_k} = \frac{1}{2}, \quad \nu_{j_l} = \frac{1}{2}, \tag{3}$$

where $i_k \in P$, $j_l \in Q$.

Shapley and Shubik (1969) noticed violent insensitivity of the core to the ratio of the dimensions of sets P and Q. In contrast, the Shapley value is sensitive to the relative scarcity of the gloves what is an attractive property of the Shapley value. The SM-nucleolus also possesses this desirable sensitivity. For the SM-nucleolus we present here the following result obtained in Tarashnina and Sharlai (2015).

Theorem 2. *Let (N, v) be a glove market game with characteristic function (1). Suppose that $|P| = p$, $|Q| = q$, $P = \{i_1, \ldots, i_p\}$, $Q = \{j_1, \ldots, j_q\}$, $p \geq q$. Then, the SM-nucleolus $\mu(v)$ is defined by the formulae*

$$\mu_{i_k} = \frac{q}{4p - 2q}, \quad \mu_{j_l} = \frac{3p - 2q}{4p - 2q}, \tag{4}$$

where $i_k \in P$ and $j_l \in Q$.

Let us introduce analytic formulae for the anti-prenucleolus in the following form.

Theorem 3. *Let (N, v) be a glove market game with characteristic function (1). Suppose that $|P| = p$, $|Q| = q$, $P = \{i_1, \ldots, i_p\}$, $Q = \{j_1, \ldots, j_q\}$, $p \geq q$. Then, the anti-prenucleolus $\nu^*(v)$ is defined by the formulae*

$$\nu^*_{i_k} = \frac{q}{2p}, \quad \nu^*_{j_l} = \frac{1}{2}. \tag{5}$$

where $i_k \in P$, $j_l \in Q$.

In order to compare the solution concepts we present a matrix A of payoffs where a_{pq} is a payoff of a right-glove holder in the game that describes the market with p right-hand owners and q left-hand ones, $p \geq 1$, $q \geq 1$.

Table 1. The prenucleolus matrix

	no. of left-glove holders			
	1	2	3	4
no. of 1	0,5	1	1	1
right- 2	0	0,5	1	1
glove 3	0	0	0,5	1
holders 4	0	0	0	0,5

As we see in Table 1 the whole payoff is given to the holders of the scarce commodity ("strong" players). The rest players receive nothing. The same outcome, that is equal to 0, the players obtain in case they do not cooperate. These players unlikely agree with that kind of allocations. In our opinion, the allocations proposed by the prenucleolus are nonviable.

From the other side, the anti-prenucleolus takes into account only the blocking power of players, and the "weak" players all together have the half of the total amount, so the strong ones do (see Table 2). That kind of divisions can be inappropriate for players with the scarce commodity.

Table 2. The anti-prenucleolus matrix

	no. of left-glove holders			
	1	2	3	4
no. of 1	0,5	0,5	0,5	0,5
right- 2	0,25	0,5	0,5	0,5
glove 3	0,167	0,333	0,5	0,5
holders 4	0,125	0,25	0,375	0,5

At the same time, the SM-nucleolus (Table 3) considers a sort of average of the prenucleolus and the anti-prenucleolus. In addition, comparing the payoffs of weak players with the payoffs the Shapley value assigns to them (Table 4) the following inequalities hold.

$$0 \leq x^w_{Pr} \leq x^w_{Sh} \leq x^w_{SM} \leq x^w_{Anti} \leq \frac{1}{2},$$

where x^w_{Pr} , x^w_{Sh}, x^w_{SM} and x^w_{Anti} are payoffs of a weak player according to the prenucleolus, the Shapley value, the SM-nucleolus and the anti-prenucleolus, correspondingly.

Table 3. The SM-nucleolus matrix

	no. of left-glove holders			
	1	2	3	4
no. of 1	0,5	0,667	0,7	0,714
right- 2	0,167	0,5	0,625	0,667
glove 3	0,1	0,25	0,5	0,6
holders 4	0,071	0,167	0,3	0,5

Table 4. The Shapley value matrix

	no. of left-glove holders			
	1	2	3	4
no. of 1	0,5	0,667	0,75	0,8
right- 2	0,167	0,5	0,65	0,733
glove 3	0,083	0,233	0,5	0,638
holders 4	0,05	0,133	0,271	0,5

In addition, we present here some significant examples to give the intuition of the considered solution concepts. The examples demonstrate some similarities and differences of the presented solutions comparing with well-known solution concepts of TU-games such as the core and the Shapley value.

Example 1. Suppose that $P = \{1, 2\}$, $Q = \{3\}$. The resulting payoff vectors for this game are presented in the following table:

The core	$C(v) = (0, 0, 1)$
The prenucleolus	$\nu(v) = (0, 0, 1)$
The Shapley value	$\varphi(v) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$
The SM-nucleolus	$\mu(v) = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$
The anti-prenucleolus	$\nu^*(v) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

The prenucleolus as well as the core assigns one to player 3 and zero to the other players. On the other hand, players 1 and 2 together can prevent player 3 from getting 1 by forming a coalition against him. Therefore, together they have the same blocking power as player 3 does.

The SM-nucleolus takes into account the blocking power of coalition $\{1, 2\}$. It assigns $\dfrac{2}{3}$ to player 3 and $\dfrac{1}{6}$ to players 1 and 2 each. Note that for this game the SM-nucleolus coincides with the Shapley value. That result was proved for an arbitrary three-person TU-game in Tarashnina, 2011. This gives some insight that the SM-nucleolus is a solution concept with similar to the Shapley value properties.

The anti-prenucleolus takes into account only the blocking power and assigns $\dfrac{1}{2}$ to player 3 and $\dfrac{1}{4}$ to players 1 and 2 each.

Example 2. Suppose that $P = \{1, 2, 3\}$, $Q = \{4\}$. The resulting payoff vectors for this game are presented in the following table:

The core	$C(v) = (0, 0, 0, 1)$
The prenucleolus	$\nu(v) = (0, 0, 0, 1)$
The Shapley value	$\varphi(v) = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{3}{4})$
The SM-nucleolus	$\mu(v) = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{7}{10})$
The anti-prenucleolus	$\nu^*(v) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$

The prenucleolus does not assigns any positive payoff to the owners of a right-hand glove, the whole amount of the total payoff goes to the holder of the unique left-hand glove. The Shapley value assigns 25 percents of the total payoff to players 1, 2, and 3 altogether, whereas the SM-nucleolus distributes 30 percents of the total payoff between the holders of a right-hand glove. The anti-prenucleolus allocates 50 percents of the total payoff between the players of different types.

Example 3. Suppose that $P = \{1, 2, 3\}$, $Q = \{4, 5\}$. The resulting payoff vectors for this game are presented in the table below.

The core	$C(v) = (0, 0, 0, 1, 1)$
The prenucleolus	$\nu(v) = (0, 0, 0, 1, 1)$
The Shapley value	$\varphi(v) = (\frac{14}{60}, \frac{14}{60}, \frac{14}{60}, \frac{39}{60}, \frac{39}{60})$
The SM-nucleolus	$\mu(v) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{5}{8}, \frac{5}{8})$
The anti-prenucleolus	$\nu^*(v) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$

In this example the Shapley value assigns 35 percents of the total payoff to players 1, 2, and 3 altogether, whereas the SM-nucleolus distributes 37,5 percents of the total amount between the holders of a right-hand glove.

As a result, we can notice that the SM-nucleolus distributes a bigger part of $v(N)$ between the players with a non-scarce glove than the Shapley value.

Finally, let us give an example when $p = q$.

Example 4. Suppose that $P = \{1, 2, 3\}$, $Q = \{4, 5, 6\}$. The resulting payoff vectors for this game are presented below.

The core	$C(v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
The prenucleolus	$\nu(v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
The Shapley value	$\varphi(v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
The SM-nucleolus	$\mu(v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
The anti-prenucleolus	$\nu^*(v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

As a matter of fact, all players in this game are treated equally and all considered solution concepts propose the same payoff vector as a solution of the game.

These examples illustrate also what happens with the right- and left-hand glove owners' payoffs according to the solutions when changing the dimensions of sets P and Q. Actually, we have demonstrated the importance of taking into account the blocking power of coalitions.

The important class of games where forming a block plays a crucial role is a class of weighted majority games.

4. The weighted majority games

The excess-based solution concepts taking into account the blocking power have an important applications in modelling the power of players in voting games. There are some well-known power indices such as the Shapley-Shubik power index and the Banzhaf index.

In such games, a proposed bill or decision is either passed or rejected. In voting body, the voting rule specifies which subsets of players are large enough to pass bills, and which are not. Those subsets that can pass bills without outside help are called winning coalitions, while those that cannot are called losing coalitions.

In such a case, we can take the worth of a winning coalition to be 1 and the worth of a losing coalition to be 0. The resulting game, in which all coalitions have a value of either 1 or 0, is called a simple game.

A simple game is completely specified once its winning coalitions are known, and it is traditional to require it to satisfy some reasonable conditions.

Definition 4. A simple game is a pair $(N, \mathcal{W})$, where N is the set of players and $\mathcal{W}$ is the collection of winning coalitions, such that

- $\emptyset \notin \mathcal{W}$ (the empty set is a losing coalition);
- $N \in \mathcal{W}$ (the grand coalition is winning);
- $S \in \mathcal{W}$ and $S \subseteq T$ imply $T \in \mathcal{W}$ (if S is a winning coalition, so is any coalition that contains S).

One common type of a simple game is a weighted voting game, which is usually represented by $[q; \omega_1, ..., \omega_n]$. Such games are defined by a characteristic function of the form

$$v(S) = \begin{cases} 1 & \text{if } \sum\limits_{i \in S} \omega_i > q, \\ \\ 0 & \text{if } \sum\limits_{i \in S} \omega_i \le q \end{cases}$$

for some non-negative numbers ω_i, called the weights, and some positive number q, called the quota. If $q = \frac{1}{2} \sum\limits_{i \in N} \omega_i$, we deal with a weighted majority game.

Example 5. Consider two versions of a voting game with four players, in which player 1 has 5 shares and players 2 to 4 have 2 shares each. The quota in the first case is 6, and in the second case is 5.

So, there are the following games [6;5,2,2,2] and [5;5,2,2,2] under consideration and we propose to look at the behavior of different solutions in these games.

The list of all coalitions and their worths are presented in Tables 5 and 7. It can be noticed that for the considered class of games the following inequality for any $S \subseteq N$ holds

$$v(S) \le w(S) \le v^*(S).$$

<u>Case 1</u>. The game [6;5,2,2,2]. Note that the core in this game consists of the unique point (1,0,0,0) and, clearly, the prenucleolus is the same point. Comparing the SM-nucleolus and the Shapley-Shubik power index one can see that the SM-nucleolus assigns to each weak player payoff $\frac{1}{10}$ what is more than the Shapley-Shubik power index does. The anti-prenucleolus treats the weak players even higher likely because it takes into account the blocking power of coalition {2,3,4}.

<u>Case 2</u>. The game [5;5,2,2,2]. This case differs from the previous one by the status of coalition {2,3,4}. Here coalition $\{2, 3, 4\}$ belongs to the set of winning coalitions of the game. This is a constant-sum simple game and it is known that the core

Table 5. Case 1. The game [6;5,2,2,2].

S	$v(S)$	$v^*(S)$	$w(S)$
1	0	1	0,5
2	0	0	0
3	0	0	0
4	0	0	0
1,2	1	1	1
1,3	1	1	1
1,4	1	1	1
2,3	0	0	0
2,4	0	0	0
3,4	0	0	0
1,2,3	1	1	1
1,2,4	1	1	1
1,3,4	1	1	1
2,3,4	0	1	0,5
1,2,3,4	1	1	1

Table 6. Case 1. Solutions of the game [6;5,2,2,2].

The prenucleolus	$\nu(v) = (1, 0, 0, 0)$
The SM-nucleolus	$\mu(v) = \left(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$
The anti-prenucleolus	$\nu^*(v) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$
The Shapley-Shubik power index	$\varphi(v) = \left(\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$

in these games is empty. Here we get $\mu(v) = \nu(v) = \nu^*(v) = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$. The difference between the these payoffs and the Shapley-Shubik payoffs are interpreted as in the previous case: the weak players get lesser in the the Shapley-Shubik power index.

However, the fact that coalition {2,3,4} is winning plays a positive role increasing the power of the weak players in two times, what cannot be said about player 1, whose power decreases.

In case 2 coalition {2,3,4} becomes winning what increases the power of the weak players in two times comparing with case 1 $\left(\frac{1}{5} \text{ in case 2 versus } \frac{1}{10} \text{ in case 1}\right)$.

5. Conclusion

The analysis of the considered games shows that for some classes of games the prenucleolus assignes inappropriate allocations to players. Then, it is necesary to consider alternative solution concepts like the SM-nucleolus and the anti-prenucleolus, which take into account the blocking power of a coalition in the game. Especially it is important for the class of weighted majority games since forming blocks there may discourage to passing a bill. The name of blocking power reflects a key point of a voting process.

Table 7. Case 2. The game [5;5,2,2,2].

S	$v(S)$	$v^*(S)$	$w(S)$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
1,2	1	1	1
1,3	1	1	1
1,4	1	1	1
2,3	0	0	0
2,4	0	0	0
3,4	0	0	0
1,2,3	1	1	1
1,2,4	1	1	1
1,3,4	1	1	1
2,3,4	1	1	1
1,2,3,4	1	1	1

Table 8. Case 2. Solutions of the game [5;5,2,2,2].

The nucleolus	$\nu(v) = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$
The SM-nucleolus	$\mu(v) = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$
The anti-prenucleolus	$\nu^*(v) = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$
The Shapley-Shubik power index	$\varphi(v) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

Apart from the solution concepts considered in the game, there is a relatively new solution concept called the α-prenucleoli set. That set consists of points, each of which takes into account the constructive power with the weight $\alpha \in [0, 1]$ and the blocking power with the weight $1 - \alpha$. Clearly, it contains the prenucleolus, the SM-nucleolus and the anti-prenucleolus. Investigation of this set-valued concept helps to find a value α for which the corresponding solution would possess good properties and be appropriate for players.

In that paper we consider the games where the players are devided on the weak and strong ones. The interesting point is to pay attention to the class of games with a veto-player or with a major player (Parilina and Sedakov, 2014; Parilina and Sedakov, 2016).

References

Aumann, R. J. and L. S. Shapley (1974). *Values of non-atomic games*. Princeton University Press.

Billera, L. J. and J. Raanan (1981). *Cores of non-atomic linear production games*. Mathematics of Operations Research, **6**, 420–423.

Britvin, S. V. and S. I. Tarashnina (2013). *Algorithms of finding the prenucleolus and the SM-nucleolus of cooperative TU-games*. Mat. Teor. Igr Prilozh., **5 (4)**, 14–32 (in Russian).

Einy, E., R. Holzman, D. Monderer and B. Shitovitz (1996). *Core and Stable Sets of Large Games Arising in Economics*. J. Economic Theory, **68**, 200–211.

Maschler, M. (1992) *The bargaining set, kernel, and nucleolus: a survey.* In: Handbook of Game Theory (Aumann, R. J. and S. Hart, eds), Vol. 1, pp. 591–665. Elsevier Science Publishers BV.

Owen, J. (1975) *On the core of linear production games.* Mathematical Programming, **9**, 358–370.

Parilina, E. and A. Sedakov (2014). *Stable Cooperation in Graph-Restricted Games*. Contributions to Game Theory and Management, **7**, 271–281.

Parilina, E. and A. Sedakov (2016). *Stable Cooperation in a Game with a Major Player.* International Game Theory Review, **18(2)**, 1640005.

Schmeidler, D. (1969). *The nucleolus of a characteristic function game.* SIAM J. Appl. Math, **17**, 1163–170.

Shapley, L. and M. Shubik (1969). *Pure competition, coalitional power, and fair division.* International Economic Review, **10(3)**, 337–362.

Smirnova, N. V. and S. I. Tarashnina (2012). *Geometrical properties of the* $[0,1]$*-nucleolus in cooperative TU-games.* Mat. Teor. Igr Prilozh., **4(1)**, 55–73 (in Russian).

Smirnova, N. V. and S. I. Tarashnina (2016). *Properties of solutions of cooperative games with transferable utilities.* Russian Mathematics, **60(6)**, 63–74.

Tarashnina, S. (2011) *The simplified modified nucleolus of a cooperative TU-game.* TOP, **19(1)**, 150–166.

Tarashnina, S. and T. Sharlai (2015). *The SM-nucleolus in glove market games.* AMS, **19(27)**, 1331–1340.