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Types of Equilibrium Points in Antagonistic Games with Ordered Outcomes

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Abstract Saddle point concept is a basic one for antagonistic games with payoff functions. For more large class consisting of games with ordered outcomes, there are different generalizations of the saddle point concept. In this article we consider three types of equilibrium for games with ordered outcomes, namely, *saddle points* (or *Nash equilibrium points*), *general equilibrium points* and *transitive equilibrium points*. The main definitions concerning games with ordered outcomes are introduced in section 1. In section 2, necessary and sufficient conditions for saddle points in games with ordered outcomes are found. These conditions are formulated by using the so-called *characteristic sets of players*. Transitive equilibrium points are considered in section 3. Theorem 3 characterizes transitive equilibrium points in antagonistic games with ordered outcomes as pre-images of saddle points in antagonistic games with payoff functions under strict homomorphisms. The main result of this article is theorem 4 in which analogy result for mixed extension of game with ordered outcomes is proved. In constructing of mixed extension of game with ordered outcomes, we use the so-called *canonical extension* of an order on the set of probabilistic measures.

Keywords: game with ordered outcomes, saddle point, general equilibrium point, transitive equilibrium point.

1. Introduction

The aim of this work is an investigation of equilibrium concept in antagonistic games with ordered outcomes. In contrast to games with payoff functions, in games with preference relations there are many types of equilibrium points. First of all we introduce the basic definitions. Formally, a game of n players with preference relations in the normal form can be given as a system of the type

$$G = \langle N, (X_i)_{i \in I}, A, (\omega_i)_{i \in I}, F \rangle \ . \tag{1}$$

where $N = \{1, \ldots, n\}$ is a set of *players*, $n \geq 2$; X_i is a set of *strategies* of the player i; A is a set of *outcomes*; $\omega_i \subseteq A^2$ is a *preference relation* for player i; F is a *realization function*, i.e. a mapping from the set of all *situations* $X = \prod_{i \in N} X_i$ into the set of outcomes A. A game G of the type (1) is called a *game with ordered outcomes* if all ω_i $(i \in N)$ are order relations.

For the class of games with ordered outcomes of the type (1), the most important optimality concept is Nash equilibrium.

Definition 1. A situation $x^0 = \left(x_i^0\right)_{i \in N}$ in the game G of the form (1) is called *Nash equilibrium point* if for all $i \in N$ and $x_i' \in X_i$ the correlation

$$F\left(x^0 \parallel x_i'\right) \overset{\omega_i}{\leq} F\left(x^0\right)$$

holds.

In the case when preference relations ω_i $(i \in N)$ not satisfy the linearity condition, we can consider a certain generalization of Nash equilibrium concept in the following manner.

Definition 2. A situation $x^0 = \left(x_i^0\right)_{i \in N}$ in game G is called *a general equilibrium point* if there does not exist $i \in N$ and $x_i' \in X_i$ such that

$$F\left(x^0 \parallel x_i'\right) \overset{\omega_i}{>} F\left(x^0\right). \tag{2}$$

An antagonistic game with ordered outcomes is a game of the type (1) in which a number of players is equal two and their preferences are mutually inverse. We consider such a game in the form

$$G = \langle X, Y, A, \omega, F \rangle \tag{3}$$

where X is a set of strategies of player 1, Y is a set of strategies of player 2, A is a set of outcomes, ω is a (partial) order relation on the set A, $F\colon X \times Y \to A$ is a realization function. The preferences of the player 1 are given by the order ω and preferences of the player 2 are given by the inverse order ω^{-1}. We assume that $|X| \geq 2$, $|Y| \geq 2$, $|A| \geq 2$. In the case the ordered set $\langle A, \omega \rangle$ is a complete lattice, the game G is called *a game with lattice-ordered outcomes*. For antagonistic game G of the form (3), the definition 1 and definition 2 have the following form.

Definition 3. A situation (x_0, y_0) in game G of the form (3) is called *Nash equilibrium point* (or *a saddle point*) if for any $x \in X, y \in Y$ hold the correlations

$$F\left(x, y_0\right) \overset{\omega}{\leq} F\left(x_0, y_0\right) \overset{\omega}{\leq} F\left(x_0, y\right). \tag{4}$$

Definition 4. A situation (x_0, y_0) in game G of the form (3) is *a general equilibrium point* if there does not exist $x \in X, y \in Y$ such that

$$F\left(x, y_0\right) \overset{\omega}{>} F\left(x_0, y_0\right) \text{ or } F\left(x_0, y\right) \overset{\omega}{<} F\left(x_0, y_0\right). \tag{5}$$

For antagonistic games with ordered outcomes, we can introduce another type of equilibrium, so-called transitive equilibrium.

Definition 5. A situation $(x_0, y_0) \in X \times Y$ is called *a transitive equilibrium point* (or briefly, *Tr-equilibrium point*) in the game G of the form (3) if there does not exist $x \in X, y \in Y$ such that

$$F\left(x, y_0\right) \overset{\omega}{>} F\left(x_0, y\right). \tag{6}$$

Remark 1. It easy to show that the following consequences

$$\text{Nash equilibrium} \Rightarrow \text{Tr equilibrium} \Rightarrow \text{General equilibrium}$$

hold and converse implications does not hold.

Particularly in antagonistic game G an arbitrary general equilibrium point need not be a transitive equilibrium point since the correlation "not more than" is not transitive (see also the example 1). A motivation for introduction of transitive equilibrium points is the fact that in game G with ordered outcomes, Tr-equilibrium points are exactly pre-images of saddle points in antagonistic games with payoff functions under strict homomorphisms of games (see Theorem 3). The main types of equilibrium points in antagonistic games with ordered outcomes are saddle points and transitive equilibrium points.

In this work we use some concepts and notations of the ordered set theory. Particularly, for arbitrary subset B of an ordered set $\langle A, \omega \rangle$ the operators $\downarrow$ and $\uparrow$ are defined as follows:

$$B^{\downarrow} = \{a \in A\colon\ (\exists b \in B)\, a \overset{\omega}{\leq} b\}, B^{\uparrow} = \{a \in A\colon\ (\exists b \in B)\, a \overset{\omega}{\geq} b\}.$$

2. Saddle points in antagonistic games

2.1. Characteristic sets of players

Consider an antagonistic game G of the form (3) with ordered outcomes.

Definition 6. We say that in the game G *an outcome $a \in A$ is guaranteed to player 1 by a strategy $x \in X$* if for any strategy $y \in Y$ the correlation $F(x, y) \overset{\omega}{\geq} a$ holds; *an outcome $a \in A$ is guaranteed to player 2 by a strategy $y \in Y$* if for any strategy $x \in X$ we have $F(x, y) \overset{\omega}{\leq} a$.

We denote by V_x^1 the set of all outcomes of game G which are guaranteed to player 1 by the strategy x and by V_y^2 the set of all outcomes guaranteed to player 2 by the strategy y i.e.

$$V_x^1 = \{a \in A\colon\ (\forall y \in Y)\, F(x, y) \overset{\omega}{\geq} a\}, V_y^2 = \{a \in A\colon\ (\forall x \in X)\, F(x, y) \overset{\omega}{\leq} a\}.$$

Definition 7. We say that in a game G *an outcome $a \in A$ is forbidden to player 1 by a strategy $y \in Y$* if for any strategy $x \in X$ the correlation $F(x, y) \overset{\omega}{\geq} a$ does not hold; *an outcome $a \in A$ is forbidden to player 2 by a strategy $x \in X$* if for any strategy $y \in Y$ the correlation $F(x, y) \overset{\omega}{\leq} a$ does not hold.

By U_y^1 we denote the set of all outcomes in game G which are non-forbidden ones for player 1 by a strategy $y \in Y$ and by U_x^2 the set of all outcomes which are non-forbidden ones for player 2 by a strategy $x \in X$:

$$U_y^1 = \{a \in A\colon\ (\exists x \in X)\, F(x, y) \overset{\omega}{\geq} a\}, U_x^2 = \{a \in A\colon\ (\exists y \in Y)\, F(x, y) \overset{\omega}{\leq} a\}. \quad (7)$$

Definition 8. An outcome $a \in A$ is called *a guaranteed outcome for player 1* if it is guaranteed at least one strategy of this player; an outcome $a \in A$ is called *a non-forbidden outcome for player 1*, if it is not forbidden to player 1 any strategy of the other player.

The set of all *guaranteed outcomes for player 1* is denoted by $V(1)$ and the set of all *non-forbidden outcomes for player 1* is denoted by $U(1)$. We have

$$V(1) = \bigcup_{x \in X} V_x^1, U(1) = \bigcap_{y \in Y} U_y^1. \quad (8)$$

For player 2 these sets are denoted by $V(2)$ and $U(2)$, respectively, and are defined dually. It follows immediately from the definitions that in any game G the inclusion $V(1) \subseteq U(1)$ holds that can be seen as analogous to the well known correlation between the lower and upper value in a game with payoff function. The dual inclusion $V(2) \subseteq U(2)$ is true also.

Definition 9. We say that a game G satisfies *the alternativeness condition* if the equality $V(1) = U(1)$ holds.

The sets of the form $V_x^1, U_y^1; V_y^2, U_x^2$ are called *characteristic sets* of player 1 and player 2, respectively. Using these sets, we define some types of optimal strategies of players in antagonistic game with ordered outcomes.

Definition 10. A strategy $x_0 \in X$ of the player 1 is called *the greatest guaranteed strategy* if it satisfies the condition $V_{x_0}^1 = V(1)$. A strategy $x_0 \in X$ of the player 1 is called *the greatest restrictive strategy* if it satisfies the condition $U_{x_0}^2 = U(2)$. For player 2, the greatest guaranteed strategy and the greatest restrictive strategy are defined dually.

Definition 11. A strategy of a player is called *a discriminating one* if it provides to penetration into the set of guaranteed outcomes of the other player. Thus, discriminating strategies $x_0 \in X$ and $y_0 \in Y$ of players 1 and 2 are characterized, respectively, by the conditions:

$$(\forall y \in Y)\, F(x_0, y) \in V(2), (\forall x \in X)\, F(x, y_0) \in V(1).$$

2.2. Necessary and sufficient conditions for saddle points

Consider an antagonistic game G with ordered outcomes of the form (3).

Theorem 1. *An arbitrary situation (x_0, y_0) in the game G is a saddle point if and only if x_0 is a discriminating strategy of player 1 and y_0 is a discriminating strategy of player 2.*

Proof (of theorem 1). Suppose a situation (x_0, y_0) is a saddle point in the game G. It is easy to show that in this case $F(x_0, y_0)$ is the greatest element in $V(1)$ and the smallest element in $V(2)$. Then we have the following equalities:

$$V(1) = (F(x_0, y_0))^{\downarrow} = \{a \in A \colon a \overset{\omega}{\leq} F(x_0, y_0)\}, \tag{9}$$

$$V(2) = (F(x_0, y_0))^{\uparrow} = \{a \in A \colon a \overset{\omega}{\geq} F(x_0, y_0)\}. \tag{10}$$

By using (9) and (10) we obtain from the definition 4 for any $x \in X$, $y \in Y$ the inclusions $F(x_0, y) \in V(2)$ and $F(x, y_0) \in V(1)$ hence the necessary condition is shown.

We now state the following supporting statement.

Lemma 1. *1) The correlation $a_1 \overset{\omega}{\leq} a_2$ holds for any $a_1 \in V(1)$ and $a_2 \in U(2)$.*
2) The correlation $b_1 \overset{\omega}{\geq} b_2$ holds for any $b_1 \in V(2)$ and $b_2 \in U(1)$.

Proof (of lemma 1). Assume $a_1 \in V(1)$ then there exists a strategy $x_0 \in X$ of player 1 such that for any $y \in Y$ the correlation $F(x_0, y) \overset{\omega}{\geq} a_1$ satisfies. The condition $a_2 \in U(2)$ means that the formula $(\forall x \in X)(\exists y \in Y) F(x, y) \overset{\omega}{\leq} a_2$ holds. Setting in this formula $x = x_0$ we obtain $F(x_0, y_0) \overset{\omega}{\leq} a_2$ for some $y = y_0$. On the other hand we have $F(x_0, y_0) \overset{\omega}{\geq} a_1$. From these two inequalities we obtain $a_1 \overset{\omega}{\leq} a_2$ and 1) is proved. Dually we have the condition 2). □

We now prove the sufficient condition in theorem 1. Assume that x_0 is a discriminating strategy of player 1 and y_0 is a discriminating strategy of player 2. Using definition 11 and the inclusion $V(2) \subseteq U(2)$ we obtain $F(x, y_0) \in V(1)$ and $F(x_0, y) \in V(2) \subseteq U(2)$ for any $x \in X$, $y \in Y$. Then in according with lemma 1 we obtain for arbitrary $x \in X$ and $y \in Y$ the correlation

$$F(x, y_0) \overset{\omega}{\leq} F(x_0, y). \tag{11}$$

Setting in (11) $x = x_0$ and then $y = y_0$ we have the following double inequality

$$F(x, y_0) \overset{\omega}{\leq} F(x_0, y_0) \overset{\omega}{\leq} F(x_0, y)$$

i.e. the situation (x_0, y_0) is a saddle point in game G. □

Theorem 2. *A situation $(x_0, y_0) \in X \times Y$ is a saddle point in game G of the form (3) if and only if*

1) *G satisfies the alternativeness condition;*
2) *x_0 is the greatest guaranteeing strategy of player 1;*
3) *y_0 is the greatest restrictive strategy of player 2.*

Lemma 2. *The inclusion*

$$V_x^1 \subseteq U_y^1 \tag{12}$$

holds for any $x \in X$, $y \in Y$.

Indeed, in accordance with definition 6, the condition $a \in V_x^1$ means that a is a general minorant for all elements of x-row then $a \overset{\omega}{\leq} F(x, y)$. Hence we obtain $a \in U_y^1$ and the inclusion (12) is shown.

Lemma 3. *A situation (x_0, y_0) is a saddle point in game G if and only if $V_{x_0}^1 = U_{y_0}^1$.*

Proof (of lemma 3). By using the operator $\downarrow$ we can write the required equality in the form

$$\bigcap_{y \in Y} (F(x_0, y))^{\downarrow} = \bigcup_{x \in X} (F(x, y_0))^{\downarrow}. \tag{13}$$

Since the conditions $a_1 \overset{\omega}{\leq} a_2$ and $a_1^{\downarrow} \subseteq a_2^{\downarrow}$ are equivalents to each other, the definition 4 of a saddle point can be presented in the form of double inclusion

$$(F(x, y_0))^{\downarrow} \subseteq (F(x_0, y_0))^{\downarrow} \subseteq (F(x_0, y))^{\downarrow} \tag{14}$$

for any $x \in X$ and $y \in Y$. Let us show that the conditions (13) and (14) are equivalents to each other. Indeed, assume that condition (14) holds. Then the subset

$(F(x_0,y_0))^{\downarrow}$ is the smallest one under inclusion between subsets $(F(x_0,y))^{\downarrow}$ $(y \in Y)$ and it is the greatest one between subsets $(F(x,y_0))^{\downarrow}$ $(x \in X)$ hence

$$\bigcap_{y \in Y} (F(x_0,y))^{\downarrow} = (F(x_0,y_0))^{\downarrow}, \ \bigcup_{x \in X} (F(x,y_0))^{\downarrow} = (F(x_0,y_0))^{\downarrow}$$

hence we obtain (13).

Conversely, suppose (13) holds. Since for any situation (x_0,y_0) the following two inclusions

$$\bigcap_{y' \in Y} (F(x_0,y'))^{\downarrow} \subseteq (F(x_0,y_0))^{\downarrow} \subseteq \bigcup_{x' \in X} (F(x',y_0))^{\downarrow} \tag{15}$$

hold, then we obtain with help (13) that the end members in (15) coincide with $(F(x_0,y_0))^{\downarrow}$. Using this fact, we have for any $x \in X$ and $y \in Y$ the following correlations:

$$(F(x,y_0))^{\downarrow} \subseteq \bigcup_{x' \in X} (F(x',y_0))^{\downarrow} = (F(x_0,y_0))^{\downarrow} = \bigcap_{y' \in Y} (F(x_0,y'))^{\downarrow} \subseteq (F(x_0,y))^{\downarrow}$$

hence (14) holds. □

By using lemmas 2 and 3, we have the following

Corollary 1. *In game G of the form (3) a saddle point there exists if and only if there exist and coincide to each other $\max\limits_{x \in X} V_x^1$ and $\min\limits_{y \in Y} U_y^1$ i.e.*

$$\max_{x \in X} V_x^1 = \min_{y \in Y} U_y^1 \tag{16}$$

(operators max *and* min *are considered with respect to inclusion). Moreover, if the left extremum is achieved at the point $x = x_0$ and the right extremum at the point $y = y_0$ then the situation (x_0,y_0) is a saddle point in game G.*

To prove the theorem 2 it remains to note that the existence of $\max\limits_{x \in X} V_x^1$ is equivalent to existence of the greatest guaranteeing strategy of player 1 and the existence of $\min\limits_{y \in Y} U_y^1$ is equivalent to the existence of the greatest restrictive strategy of player 2; moreover it follows from (16) that the game G satisfies the alternativeness condition which completes the proof of theorem 2.

Corollary 2. *For antagonistic games with lattice-ordered outcomes, the equality (16) takes the form*

$$\max_{x \in X} \inf_{y \in Y} F(x,y) = \min_{y \in Y} \sup_{x \in X} F(x,y)$$

and it coincides with well known condition for the existence of a saddle point in antagonistic game with payoff function (see, for example, Vorob'ev, 1985).

3. Transitive equilibrium points in antagonistic games

3.1. Connection between transitive equilibrium points and saddle points

Consider an antagonistic game G with ordered outcomes of the form (3). Let $\varphi\colon A \to \mathbb{R}$ be a function from the set A of outcomes of game G into real numbers. Then we can construct the following antagonistic game with payoff function

$$G_\varphi = \langle X, Y, \varphi \circ F \rangle. \tag{17}$$

Theorem 3. *Let G be an antagonistic game with ordered outcomes of the form (3). Then*

1. If a situation $(x_0, y_0) \in X \times Y$ is a saddle point in game G_φ where $\varphi\colon A \to \mathbb{R}$ is some strict isotonic function from the set of outcomes A into $\mathbb{R}$ then (x_0, y_0) is a transitive equilibrium point in game G.

In the case when the set of outcomes A is finite or countable the converse is truth also, namely we have the following statement.

2. If a situation $(x_0, y_0) \in X \times Y$ is a transitive equilibrium point in game G, then there exists a strict isotonic function $\varphi\colon A \to \mathbb{R}$ from the set of outcomes A into $\mathbb{R}$ such then (x_0, y_0) is a saddle point in the game G_φ.

Remark 2. The assertion 2 becomes false when replacing "transitive equilibrium point" by "general equilibrium point" (see example 1).

Proof (of theorem 3). 1. Suppose that the situation (x_0, y_0) is not a transitive equilibrium point in game G, i.e. there exist the strategies $x_1 \in X$ and $y_1 \in Y$ such that $F(x_1, y_0) \overset{\omega}{>} F(x_0, y_1)$. Because the function φ is strict isotonic one, we obtain $\varphi(F(x_1, y_0)) > \varphi(F(x_0, y_1))$ in contradiction with our assumption that (x_0, y_0) is a saddle point in the game G_φ. To prove the assertion 2, we need in the following lemmas.

Lemma 4. *(see Rozen, 1988). Consider an arbitrary ordered set $\langle A, \omega \rangle$ and $B \subseteq A$, $C \subseteq A$. Assume that $\neg \left(b \overset{\omega}{>} c \right)$ for any $b \in B$ and $c \in C$. Then there exists a linear co-ordering $\overline{\omega}$ of the order ω such that $b \overset{\overline{\omega}}{<} a \overset{\overline{\omega}}{<} c$ for any $b \in B \setminus C$, $a \in B \cap C$, $c \in C \setminus B$.*

Lemma 5. *(see Rozen, 1988). Assume that $\neg \left(b \overset{\omega}{>} c \right)$ for any $b \in B, c \in C$. Then there exists a strict isotonic function $\varphi_0\colon A \to \mathbb{R}$ such that $\varphi_0(b) \leq \varphi_0(c)$ for any $b \in B$ and $c \in C$* (this function is called *a separating function*).

We now prove the theorem 3. Let a situation $(x_0, y_0) \in X \times Y$ be a transitive equilibrium point in game G. Put $B = \{F(x, y_0) : x \in X\}$, $C = \{F(x_0, y) : y \in Y\}$. Then the condition $\neg \left(b \overset{\omega}{>} c \right)$ for any $b \in B, c \in C$ holds and using lemma 5 we obtain that there exists a strict isotonic function $\varphi\colon A \to \mathbb{R}$ such that $\varphi(F(x, y_0)) \leq \varphi(F(x_0, y))$ for any $x \in X$ and $y \in Y$. It follows that the situation (x_0, y_0) is a saddle point in game G_φ. $\square$

Example 1. Consider an antagonistic game G with ordered outcomes in which realization function F by the Table 1 and the order relation ω by its diagram (see Fig. 1 are given.

Table 1. Realization function F

F	y_0	y_1
x_0	c	b
x_1	d	a

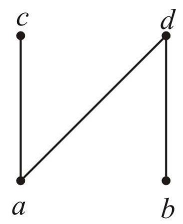

Fig. 1. Order relation ω

In this game the situation (x_0, y_0) *is a general equilibrium point* (since the element $F(x_0, y_0) = c$ is a minimal one in its row and it is a maximal one in its column) but the situation (x_0, y_0) *is not a transitive equilibrium point* (since the correlations $F(x_1, y_0) = d \overset{\omega}{>} b = F(x_0, y_1)$ hold). We now show without using of theorem 3 that there does not exist a strict isotonic function $\varphi \colon A \to \mathbb{R}$ under which the situation (x_0, y_0) is a saddle point in the game G_φ. Indeed in this case we have the following double inequality for any $x \in X$ and $y \in Y$:

$$\varphi(F(x, y_0)) \leq \varphi(F(x_0, y_0)) \leq \varphi(F(x_0, y)) . \tag{18}$$

Then setting in (18) $x = x_1$ and then $y = y_1$ we obtain

$$\varphi(F(x_1, y_0)) \leq \varphi(F(x_0, y_0)) \leq \varphi(F(x_0, y_1))$$

hence $\varphi(F(x_1, y_0)) \leq \varphi(F(x_0, y_1))$ that is $\varphi(d) \leq \varphi(b)$. On the other hand since the function φ is strict isotonic, the condition $d \overset{\omega}{>} b$ implies $\varphi(d) > \varphi(b)$ in contradiction with above inequality.

3.2. A mixed extension of an antagonistic game with ordered outcomes

The main result of the section 3 is a description of the set of transitive equilibrium points in mixed extension of an antagonistic game with ordered outcomes. First of all we need the following notations. For arbitrary ordered set $\langle A, \omega \rangle$ we denote by $C_0(\omega)$ the set of all isotonic function from $\langle A, \omega \rangle$ into real numbers $\mathbb{R}$ and by $C(\omega)$ the set of all strict isotonic function from $\langle A, \omega \rangle$ into real numbers. By *a probabilistic measure* on a finite ordered set $\langle A, \omega \rangle$ we shall mean a non-negative function $p \colon A \to \mathbb{R}$ such that $\sum\limits_{a \in A} p(a) = 1$. The set of all probabilistic measures on arbitrary set A is denoted by $\widetilde{A}$. For any $\varphi \in C_0(\omega)$ and $p \in \widetilde{A}$ we put $\overline{\varphi}(p) = (\varphi, p)$ where (φ, p) is the standard scalar product.

Definition 12. Let $\langle A, \omega\rangle$ be a finite ordered set. *The canonical extension of the order ω on the set of probabilistic measures* is called a binary relation $\widetilde{\omega} \subseteq \widetilde{A} \times \widetilde{A}$ defined by the formula:

$$p_1 \overset{\widetilde{\omega}}{\leq} p_2 \Leftrightarrow (\forall \varphi \in C_0(\omega)))\, \overline{\varphi}(p_1) \leq \overline{\varphi}(p_2)\,.$$

It is known that $\widetilde{\omega}$ is an order relation on the set $\widetilde{A}$ of probabilistic measures.

In an evident form, the order relation $\widetilde{\omega}$ can be presented as follows. Put for arbitrary probabilistic measure $p \in \widetilde{A}$ and for arbitrary subset $B \subseteq A$: $p(B) = \sum\limits_{a \in B} p(a)$. Then we have the following equivalence

$$p_1 \overset{\widetilde{\omega}}{\leq} p_2 \Leftrightarrow (\forall B \in M(\omega)))\, p_1(B) \leq p_2(B) \tag{19}$$

where $M(\omega)$ is a family of all majorant stable subsets in the ordered set $\langle A, \omega\rangle$ (note the subset $B \subseteq A$ is called majorant stable if conditions $a \in B$ and $a' \overset{\omega}{>} a$ imply $a' \in B$).

Definition 13. *By the mixed extension of a finite antagonistic game*

$$G = \langle X, Y, A, \omega, F\rangle$$

with ordered outcomes we mean an antagonistic game with ordered outcomes of the form

$$\widetilde{G} = \langle \widetilde{X}, \widetilde{Y}, \widetilde{A}, \widetilde{\omega}, \widetilde{F}\rangle \tag{20}$$

where $\widetilde{X}$ is the set of probabilistic measures on X, $\widetilde{Y}$ the set of probabilistic measures on Y, $\widetilde{A}$ the set of probabilistic measures on A, $\widetilde{\omega}$ is the canonical extension of order ω on the set of probabilistic measures and $\widetilde{F}$ is a mapping from the set $\widetilde{X} \times \widetilde{Y}$ into $\widetilde{A}$ which is defined as follows. For any probabilistic measures $\mu \in \widetilde{X}$ and $\nu \in \widetilde{Y}$ we set $\widetilde{F}(\mu, \nu) = F_{(\mu,\nu)}$ where $F_{(\mu,\nu)}$ is a probabilistic measures on A which is given by the equality

$$F_{(\mu,\nu)}(a) = \sum_{F(x,y)=a} \mu(x)\, \nu(y)\,. \tag{21}$$

Thus the transition from an antagonistic game with ordered outcomes to its mixed extension means to replace the basic sets by sets of probability measures and extension of the order relation and the realization function.

3.3. Transitive equilibrium points in mixed extension of antagonistic game with ordered outcomes

Theorem 4. *Consider a finite antagonistic game G with ordered outcomes and let $\widetilde{G}$ be its mixed extension. An arbitrary situation in mixed strategies $(\mu_0, \nu_0) \in \widetilde{X} \times \widetilde{Y}$ is a transitive equilibrium point in game $\widetilde{G}$ if and only if there exists a strict isotonic function $\varphi \in C(\omega)$ from the set of outcomes A into real numbers $\mathbb{R}$ such that the situation (μ_0, ν_0) is a saddle point in the mixed extension (in the classical sense) of the antagonistic game $G_\varphi = \langle X, Y, \varphi \circ F\rangle$ with payoff function.*

The proof of "if part" is based on the following lemmas (see Rozen, 2014).

Lemma 6. *Let $\langle A,\omega\rangle$ be an arbitrary finite ordered set and φ be a strict isotonic mapping from $\langle A,\omega\rangle$ into real numbers $\mathbb{R}$. Define an extension $\overline{\varphi}$ of the function φ on the set $\widetilde{A}$ setting for any $p\in\widetilde{A}$*

$$\overline{\varphi}\,(p)=\sum_{a\in A}p\,(a)\,\varphi\,(a)\,. \tag{22}$$

Then $\overline{\varphi}$ is a strict isotonic mapping from the ordered set $\langle\widetilde{A},\widetilde{\omega}\rangle$ into real numbers $\mathbb{R}$.

Lemma 7. *Given a finite antagonistic game $G=\langle X,Y,A,\omega,F\rangle$ with ordered outcomes and an arbitrary isotonic function $\varphi\in C_0\,(\omega)$, we can construct an antagonistic game $G_\varphi=\langle X,Y,\varphi\circ F\rangle$ with payoff function. Let $\overline{\varphi\circ F}$ be the payoff function in the mixed extension (in the classical sense) of game G_φ. Then for any situation in mixed strategies $(\mu,\nu)\in\widetilde{X}\times\widetilde{Y}$ the following equality*

$$\overline{\varphi\circ F}\,(\mu,\nu)=\overline{\varphi}\left(F_{(\mu,\nu)}\right) \tag{23}$$

holds where the probabilistic measure $F_{(\mu,\nu)}$ by (21) and the function $\overline{\varphi}$ by (22) are defined.

We now prove the "if part" in theorem 4. Indeed, suppose that there exists a function $\varphi\in C\,(\omega)$ such that a situation (μ_0,ν_0) is a saddle point in the mixed extension (in the classical sense) of antagonistic game $G_\varphi=\langle X,Y,\varphi\circ F\rangle$ with payoff function. We need to show that (μ_0,ν_0) is a transitive equilibrium point in game $\widetilde{G}$. Otherwise we have the correlation

$$F_{(\mu_1,\nu_0)}\overset{\widetilde{\omega}}{>}F_{(\mu_0,\nu_1)} \tag{24}$$

for some $\mu_1\in\widetilde{X}$ and $\nu_1\in\widetilde{Y}$. By using lemma 6 we obtain from (24) $\overline{\varphi}\left(F_{(\mu_1,\nu_0)}\right)>\overline{\varphi}\left(F_{(\mu_0,\nu_1)}\right)$; it can be written with accordance to lemma 7 in the form

$$\overline{\varphi\circ F}\,(\mu_1,\nu_0)>\overline{\varphi\circ F}\,(\mu_0,\nu_1)\,. \tag{25}$$

On the other hand, since the situation (μ_0,ν_0) is a saddle point in the mixed extension of antagonistic game G_φ, we have

$$\overline{\varphi\circ F}\,(\mu_1,\nu_0)\le\overline{\varphi\circ F}\,(\mu_0,\nu_0)\le\overline{\varphi\circ F}\,(\mu_0,\nu_1)$$

hence $\overline{\varphi\circ F}\,(\mu_1,\nu_0)\le\overline{\varphi\circ F}\,(\mu_0,\nu_1)$ that contradict to (25). Thus the "if part" in theorem 4 is proved. To prove the converse, we need the following

Lemma 8. *Let $\langle A,\omega\rangle$ be an arbitrary finite ordered set and $P,Q\subseteq\widetilde{A}$ be two polyhedrons of probabilistic measures. Assume that for any $p\in P,q\in Q$ the condition $\neg\left(p\overset{\widetilde{\omega}}{>}q\right)$ holds. Then there exists a strict isotonic mapping $\varphi\in C\,(\omega)$ from the ordered set $\langle A,\omega\rangle$ into real numbers such that $\overline{\varphi}\,(p)\le\overline{\varphi}\,(q)$ for any $p\in P,q\in Q$.*

Proof (of lemma 8). Let $(A_1,\ldots,A_m)$ be a list of all majorant stable subsets in the ordered set $\langle A,\omega\rangle$. Consider a mapping ξ which to every probability measure $p\in\widetilde{A}$ put in correspondence the vector $\xi\,(p)=(p\,(A_1)\,,\ldots,p\,(A_m))\in\mathbb{R}^m$. Since the mapping ξ is a linear one it translates P and Q in convex polyhedrons $\xi(P)$ and $\xi(Q)$ respectively. Then $R = \xi(P) + (-1)\xi(Q)$ be a convex polyhedron in $\mathbb{R}^m$ (see, for example, Leichtweiss, 1980). Show that the polyhedron R does not contain of semi-positive vectors. Otherwise, assume that there exists a vector $u = (u_1, \ldots, u_m) \in R$ provided $u_1 \geq 0, \ldots, u_m \geq 0$ and at least one of these inequalities is strict. In this case there exist vectors $p \in P$ and $q \in Q$ such that

$$(u_1, \ldots, u_m) = (p(A_1), \ldots, p(A_m)) - (q(A_1), \ldots, q(A_m)),$$

i.e. $p(A_i) - q(A_i) = u_i \geq 0$ for all $i = 1, \ldots, m$ and at least one inequality is strict.

By using (19), we obtain $p \overset{\tilde{\omega}}{>} q$ that contradict to our assumption. Thus the convex polyhedron R does not contain of semi-positive vectors. Then for this polyhedron R there exists a hyperplane of support with strict positive normal vector $c \in \mathbb{R}^m$, which contains the null vector $\mathbf{0} \in \mathbb{R}^m$ (see Rozen, 2014, lemma IV.7), i.e. $(\xi(p) - \xi(q), c) \leq 0$ for any $p \in P, q \in Q$, hence $(c, \xi(p)) \leq (c, \xi(q))$. For conjugate liner mapping ξ^* we have $(\xi^*(c), p) \leq (\xi^*(c), q)$ for any $p \in P$, $q \in Q$. It remains to note that in the case vector $c \in \mathbb{R}^m$ is a positive one, $\xi^*(c)$ is a strict isotonic mapping from the ordered set $\langle A, \omega \rangle$ into real numbers $\mathbb{R}$ (see Rozen, 2014, lemma IV.10) $\square$

Let us prove the "only if part" in theorem 4. Assume that a situation $(\mu_0, \nu_0) \in \widetilde{X} \times \widetilde{Y}$ in mixed strategies is a transitive equilibrium point in a game $\widetilde{G}$ of the form (20), then for all $\mu \in \widetilde{X}$ and $\nu \in \widetilde{Y}$ we have

$$\neg \left(F_{(\mu,\nu_0)} \overset{\tilde{\omega}}{>} F_{(\mu_0,\nu)} \right). \tag{26}$$

Put $P = \{F_{(\mu,\nu_0)} \colon \mu \in \widetilde{X}\}$, $Q = \{F_{(\mu_0,\nu)} \colon \nu \in \widetilde{Y}\}$. It is easy to check that the set P coincides with the convex hull of the finite set $\{F_{(x,\nu_0)} \colon x \in X\}$ and the set Q coincides with the convex hull of the finite set $\{F_{(\mu_0,y)} \colon y \in Y\}$, hence P and Q are convex polyhedrons. Moreover, it follows directly from (26) that the condition $\neg \left(p \overset{\tilde{\omega}}{>} q \right)$ holds for any $p \in P$, $q \in Q$. Thus, all assumptions of lemma 8 satisfy here. According with lemma 8 there exists a strict isotonic mapping $\varphi \in C(\omega)$ from the ordered set $\langle A, \omega \rangle$ into real numbers such that the inequalities

$$\overline{\varphi}\left(F_{(\mu,\nu_0)}\right) \leq \overline{\varphi}\left(F_{(\mu_0,\nu)}\right) \tag{27}$$

hold for any $\mu \in \widetilde{X}$ and $\nu \in \widetilde{Y}$. By using lemma 7, we can write (27) in the form

$$\overline{\varphi \circ F}(\mu, \nu_0) \leq \overline{\varphi \circ F}(\mu_0, \nu) \tag{28}$$

where $\overline{\varphi \circ F}$ is the payoff function in the mixed extension (in the classical sense) of game G_φ. Setting in (28) $\mu = \mu_0$ and then $\nu = \nu_0$ we obtain

$$\overline{\varphi \circ F}(\mu, \nu_0) \leq \overline{\varphi \circ F}(\mu_0, \nu_0) \leq \overline{\varphi \circ F}(\mu_0, \nu)$$

i.e. the situation (μ_0, ν_0) is a saddle point in the mixed extension of game G_φ which completes a proof of theorem 4.

Corollary 3. *The set $TrEq\ \widetilde{G}$ of transitive equilibrium points it the mixed extension of game G can be presented in the form*

$$TrEq\ \widetilde{G} = \bigcup_{\varphi \in C(\omega)} Sp\ \overline{G}_\varphi$$

where $Sp\ \overline{G}_\varphi$ is the set of saddle points in mixed extension (in the classical sense) of game G_φ and the function φ runs over the set $C(\omega)$ consisting of all strict isotonic mappings of the ordered set $\langle A, \omega \rangle$ into real numbers.

Remark 3. The statement of theorem 4 is not a consequence of a description of equilibrium points in mixed extension of games with ordered outcomes which was given in (Rozen, 2010). Particularly, assume a situation (μ_0, ν_0) in mixed extension of antagonistic game G with ordered outcome is a general equilibrium point but is not a transitive equilibrium point. Then there exists two strict isotonic functions $\varphi \in C(\omega)$ and $\psi \in C(\omega^{-1})$ such that the situation (μ_0, ν_0) is Nash equilibrium point in mixed extension (in the classical sense) of game $\langle X, Y, \varphi \circ F, \psi \circ F \rangle$ with payoff functions. However in this case there does not exist one strict isotonic function $\varphi \in C(\omega)$ such that the situation (μ_0, ν_0) is a saddle point in mixed extension (in the classical sense) of antagonistic game $G_\varphi = \langle X, Y, \varphi \circ F \rangle$ (see also the example 1).

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