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Game-Theoretic Approach for Modeling of Selfish and Group Routing

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Abstract The development of methodological tools for modeling of traffic flow assignment is crucial issue since traffic conditions influence significantly on quality of life nowadays. Herewith no secret that the development of in-vehicle route guidance and information systems could impact significantly on route choice as soon as it is highly believed that they are able to reduce congestion in an urban traffic area. Networks' users join groups of drivers who rely on the same route guidance system. Therefore, present paper is devoted to discussing approaches for modeling selfish and group routing. Network performance is deeply associated with competition between users of networks. So, the emphasis in our discussion is placed on game-theoretic approaches for appropriate modeling.

Keywords: traffic assignment problem, selfish routing, user equilibrium of Wardrop, group routing, Nash equilibrium, system optimum of Wardrop.

1. Introduction

J.G. Wardrop published in 1952 two principles for traffis flow assignment (Wardrop, 1952). The first principle of Wardrop associated with user equilibrium assignment states:

- The journey time on all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Clear, that it is still relevant and useful for evaluation of traffic flow assignment. It means, that traffic engineers and decision makers at the different levels of management rely on first principle to identify used routes in the network and the appropriate flows. The second principle of Wardrop associated with system optimum:

- The average journey time is a minimum.

The second principle could not be used for evaluation of traffic flow assignment since no one driver seeks to minimize average time. Any driver tries to minimize its own travel time and, hence, the first principle of Wardrop is more appropriate for purpose of such evaluation. Nevertheless, the relationships between user equilibrium and system optimum define important economical conclusions in transportation about toll pricing on the links of network (Gartner, 1980). Eventually, due to the second Wardrop's principle it is possible to evaluate toll prices that could be charged from users of a road network.

There exist a number of researches that dealt with these two principles of Wardrop. The extensive review of existing models and methods implemented for traffic assignment evaluation is made by M. Patriksson in 1994 (Patriksson, 1994). This outstanding book was republished in the beginning of 2015 without loss of its relevance (Patriksson, 2015). In spite of various results already obtained in this branch of applied mathematics, new investigations are still appearing to contribute namely practically expanding numerical techniques. For instance, the past decade much attention has been paid to a class of bush-based algorithms (Zheng and Peeta, 2014). According to (Xie et al., 2013), it is the ability of such algorithms to solve large-scale traffic assignment problems at a high level of precision that attracts many researchers. On the other hand, there have been a relatively weak development of theoretical principles concerning traffic assignment problem since the early 1950s.

The important substantive insights on the first Wardrop's principle were made by T. Roughgarden in his book (Roughgarden, 2005). He deeply investigated user equilibrium concept as selfish behaviour of network's users. However, the certain relationships between selfish routing and group behavior in a network (system optimum) were not established. Nevertheless, it was clearly shown, how selfish routing could be inefficient from system perspective. In other words, when each driver tries to minimize its own travel time (selfish routing), then the average traffic conditions deteriorate. Much attention was payed to Braess's paradox and its extension on second and third Braess graphs. The task to avoid this paradox seems unsolvable for large road networks. From practical perspective, any topology of real road network is really exposed to the manifestation of Braess's paradox. Therefore, selfish routing in real road network lead to the loss of efficiency of network performance. The only way to increase network performance is developing of the group routing principle (Krylatov et al., 2016).

A number of theoretical results that establish relationships between user equilibrium of Wardrop, Nash equilibrium (competition between several groups of users) and system optimum of Wardrop were obtained in (Krylatov et al., 2016). Investigation was based on assumption that the impact of in-vehicle route guidance and information (IVRGI) systems on route choice in daily trips of people increases nowadays. Such systems are result of the rapid development of information technologies in the past three decades leads by the way to the emergence of different specialized telecommunication systems, which nowadays are introduced almost in every field of human activity. The influence of these systems on decision making seems to be significant. Moreover, the permanent innovative development of such systems is noticeably related to the creation of intelligent vehicles. Indeed, from a consumer perspective one of the main attributes of any intelligent vehicle is an automatic drive regime that is associated with an automatic in-vehicle route guidance system. Therefore, guidance systems are seemed to be an integral part of the concept of intelligent vehicle.

Actually, an automatic guidance system is a great advantage of intelligent vehicles not only from a consumer perspective. The traffic flow of intelligent vehicles could be automatically assigned by the central guidance system in such a way to minimize overall travel time of all road network users. In other words, a *system optimum* (Wardrop, 1952) could be reached on the network by imposing the optimal route choices on the users (Patriksson, 2015; Sheffi, 1985; Wardrop, 1952). Such an assignment of traffic flow is often called *involuntary system optimum*, unlike *voluntary system optimum* (Gartner, 1980). In a voluntary system optimum case, after paying special charging tolls users reached system optimum, although initially they were tending to *user equilibrium* assignment (Patriksson, 2015; Sheffi, 1985; Wardrop, 1952). Here, it should be mentioned that user equilibrium assignment is supposed to take place when all drivers tries to minimize their own travel time without support of any guidance system. As a result, the overall travel time spending by all users assigned according to interest of each atomic user is more than overall travel time in system optimum case. Therefore, central guidance system is capable to reduce congestions by imposing system optimum assignment on the intelligent vehicles.

Moreover, according to (Bonsall, 1992) there is considerable government interest in the development of in-vehicle guidance systems. This interest reflects a belief that such systems could produce benefits in four ways (Bonsall, 1992):

- to improve people's knowledge of the network and assist them to find efficient routes;
- to reduce unnecessary mileage, traffic volumes, and hence congestion;
- to link in-vehicle guidance system with traffic control and, perhaps, road pricing systems;
- to obtain more globally efficient routing patterns (system optimum).

Therefore governments feel possible positive effects from implementation of in-vehicle guidance systems and try to formalize them. From the mathematical side all these advantages could be expressed by system optimum principle. Hence, all mentioned ways of producing benefits to the traffic systems are already discussed in the previous paragraph in a short form of the "optimizational vocabulary".

Despite the interest of government, as a rule the major contribution in development of guidance systems are produced by different private business companies. By the virtue of competitive structure of economics these companies are forced to compete with each other offering their own users better service. First of all, "better service" means the less travel time from origin to destination point. Thus, each company seeks to route the flows of its own users so to minimize their average travel time. At the same time others try to minimize average travel time of their users routing in the same road network. Due to described circumstances the *competitive traffic assignment* problem is appeared. Non-cooperative nature of relations between the companies leads to the set of such optimization programs that the unknown variables of any of these programs are independent parameters in all others. Therefore, competitive traffic assignment problem should be formulated in a game theoretic form with Nash equilibrium search (Nash, 1951).

System optimum assignment obtained by in-vehicle guidance systems with one provider is assumed to differ from assignment imposed by Nash equilibrium strategies of the competitive companies offered their consumers route guidance. Thus, investigation of relationships between Wardrop's system optimum associated with traffic assignment problem and Nash equilibrium associated with competitive traffic assignment problem seems quite important. Indeed, when flows of intelligent vehicles are large enough then competitive guidance systems could deviate traffic assignment from system optimum significantly. This fact should be taken into consideration by traffic engineers, transportation planners, network designers and etc. in transportation modeling. This paper is completely focused on mentioned relationships and moreover, some common aspects of Nash equilibrium and user equilibrium of Wardrop assignments will be also illuminated.

2. State-of-The-Art

We give here an overview of the results, obtained in sphere of traffic behavior modeling by virtue of game-theoretical approach. This is quite a standard overview of the results on the topic (one could see (Krylatov et al., 2016; Patriksson, 2015)).

The first attempt to define traffic equilibria in forms of network games was made in the late 1950s by Charnes and Cooper (Charnes and Cooper, 1958). They described the user equilibrium flow as a non-cooperative Nash equilibrium in a game where the players are pairs of origin–destination (OD pairs), competing to minimize travel time of their respective commodity flows. Further discussion along this line is developed by Dafermos in (Dafermos, 1971; Dafermos and Sparrow, 1969). However, these first investigations of the relationships between a Wardrop equilibrium and a network games have not driven to any formal expressions.

Rosenthal studied a discrete version of the user equilibrium traffic assignment problem in 1973 (Rosenthal, 1973). It should be stressed that the players are defined as the individual travelers, with strategy spaces equal to their respective sets of routes available. Travelers seeking to minimize their individual travel time, i.e. their payoff functions. The game is shown to be equivalent to a non-cooperative, pure-strategy Nash game in the traffic network. Therefore, he was the first one who formulated special case of competitive traffic assignment problem as we defined it above. This is the special case since each UG consist of solely one user.

Devarajan in 1981 extended discrete version to the continuous case, however, as do Charnes and Cooper, defines OD pairs as the players (Devarajan, 1981) and consider the payoff functions:

$$\varphi_w(\mathbf{y}) = \sum_{a\in A_w} \int_0^{y_a} t_a(s)ds, \quad \forall w \in W,$$

where W is a set of UGs, $w \in W$; A_w is a set of links included in routes between OD pair w; y_a is a traffic flow on a congested link a; t_a is a travel time through a congested link a. Hence his formulation is not correspond to competitive traffic assignment principle. Nevertheless, it was proved that the Nash game thus defined is equivalent to a Wardrop equilibrium.

In the middle of 1980s more general game formulations of traffic equilibria were given by Fisk (Fisk, 1984) and Haurie and Marcotte (Haurie and Marcotte, 1985). The travel made in an OD pair is divided into a number of players and, hence, a player, as defined, may use several routes simultaneously; in equilibrium, all players divide their flow on all routes used in the OD pair. Therefore, such formulation has the certain common features with competitive traffic assignment problem. However, it is shown that only in the limiting case, when the number of players in each OD pair tends to infinity, while sharing the same strategy, the Nash game is equivalent to a Wardrop equilibrium.

In the 1990 the development of computer networks motivated researchers to begin investigation of competitive routing in multiuser communication networks (Orda et al., 1993). According to Orda *et al.*, a single administrative domain was no longer a valid assumption in networking. Then communication networks shared by selfish users with their own given flow demands were considered and modeled as noncooperative games by several research groups (Korilis and Lazar, 1995; Korilis et al., 1995; La and Anantharam, 1997) and (Altman et al., 2002). Due to these researches different properties of such systems are established, and the conditions of existence and uniqueness of Nash equilibrium in multiuser communication networks are widely studied.

During the 2000s Altman *et al.* have extended results, obtained for multiuser communication networks, and then implemented them to road networks (Altman et al., 2002; Altman et al., 2011; Altman and Kameda, 2005). Unlike Haurie and Marcotte, Altman *et al.* established the convergence of the Nash equilibrium in network games to the Wardrop equilibrium as the number of players grows under weaker convexity assumptions (Altman et al., 2011). Therefore, they raised again the question of relationships between Nash equilibrium in noncooperative n-person network games and Wardrop equilibrium in the traffic assignment problem.

3. Traffic assignment problem

This section is devoted to the basic description of the modern traffic assignment problem.

Consider a transportation network presented by oriented graph $G = (N, A)$. We assume, that there is a set of OD pairs W and the sets of routes R^w between each OD pair $w \in W$. Moreover, introduce following notation: F^w is the demand between w; f_r^w is the flow of UG j through $r \in R^w$; x_a is the flow through the arc $a \in A$, $x = (\ldots, x_a, \ldots)$; $t_a(x) = t_a(x_a)$ is the travel time of flow x_a through congested arc $a \in A$; $\delta_{a,r}^w$ is an indicator:

$$\delta_{a,r}^w = \begin{cases} 1, \text{ if route } r \in R^w \text{ includes arc } a \in A, \\ 0, \text{ otherwise.} \end{cases}$$

According to (Beckmann et al, 1956; Patriksson, 1994; Sheffi, 1985), the equal travel time on all actually used routes, that is less than travel time on any unused route, could be reached by assignment strategy obtained from the following optimization program:

$$\mathbf{x}^{ue} = \arg\min_x \sum_{a \in A} \int_0^{x_a} t_a(u)du, \tag{1}$$

subject to

$$\sum_{r \in R^w} f_r^w = F^w \quad \forall w \in W, \tag{2}$$

$$f_r^w \geq 0 \quad \forall r \in R^w, w \in W, \tag{3}$$

with definitional constraints

$$x_a = \sum_{w \in W} \sum_{r \in R^w} f_r^w \delta_{a,r}^w \quad \forall a \in A. \tag{4}$$

Let us offer the basic theorem and its proof for user equilibrium of Wardrop (Patriksson, 2015; Sheffi, 1985). The proof of the following theorem has an important meaningful sense.

Theorem 1. *Solution x^* of optimization problem (1)–(4) is user equilibrium of Wardrop.*

Proof. Lagrangian of (1)–(4) is

$$L = \sum_{a \in A} \int_0^{x_a} t_a(u)du + t^w \left(F^w - \sum_{r \in R_{rs}} f_r^w \right) + \sum_{r \in R_w} \left(-f_r^w\right) \eta_r^w,$$

where t^w and $\eta_r^w \geq 0$, $r \in R^w$, $w \in W$ are multipliers of Lagrange. According to Kuhn–Tucker conditions, partial derivatives of L in x^* are equal to zero:

$$\frac{\partial L}{\partial f_r^w} = \sum_{a \in A} t_a(x_a) \cdot \frac{\partial x_a}{\partial f_r^w} - t^w - f_r^w = 0 \quad \forall r \in R^w, w \in W. \tag{5}$$

Due to (4):

$$\frac{\partial x_a}{\partial f_r^w} = \delta_{a,r}^w \quad \forall a \in A, r \in R^w, w \in W.,$$

that leads from (5) to

$$\frac{\partial L}{\partial f_r^w} = \sum_{a \in A} t_a(x_a) \cdot \delta_{a,r}^w - t^w - f_r^w = 0 \quad \forall r \in R^w, w \in W. \tag{6}$$

Complementary slackness require $f_r^w \cdot \eta_r^w = 0$. If $f_r^w > 0$, then $\eta_r^w = 0$, and, if $f_r^w = 0$, then $\eta_r^w \geq 0$. Than we obtain:

$$\sum_{a \in A} t_a(x_a) \cdot \delta_{a,r}^w \begin{cases} = t^w, \text{ if } f_r^w > 0 \\ \geq t^w, \text{ if } f_r^w = 0 \end{cases} \quad \forall \ r \in R^w. \tag{7}$$

Therefore, solution of (1)–(4) fulfills (7). Consequently, it is user equilibrium of Wardrop by definition.

Note, that since goal function (1) has no any physical or economical scence, expression (5) is crucial for selfish routing. It confirms that problem (1)–(4) is the behavioral model of selfish routing when each user tries to minimize its own travel time.

According to (Beckmann et al, 1956; Patriksson, 1994; Sheffi, 1985), the minimum average travel time could be reached by assignment strategy obtained from the following optimization program:

$$T(\mathbf{x}^{so}) = \min_x \sum_{a \in A} t_a(x_a) x_a, \tag{8}$$

subject to

$$\sum_{r \in R^w} f_r^w = F^w \quad \forall w \in W, \tag{9}$$

$$f_r^w \geq 0 \quad \forall r \in R^w, w \in W, \tag{10}$$

with definitional constraints

$$x_a = \sum_{w \in W} \sum_{r \in R^w} f_r^w \delta_{a,r}^w \quad \forall a \in A. \tag{11}$$

Unlike user equilibrium model, problem (8)–(11) is quite clear as soon as goal function (8) has an accurate economical scence (total time costs).

4. Game of OD-pairs

This section is devoted to the alternative game-theoretic formulation of user-equilibrium assignment. All discussed here results are obtained by S. Devarajan (Devarajan, 1981).

Assume that OD-pairs are players. Then we define $A_w \subset A$ as subset of arcs that are used by traffic flows between pair $w \in W$: $A_w = \{a : a \in r, \text{ for some } r \in R^w\}$. Each player w tries to minimize its payoff function

$$P_w(x) = P_w(x^w, x^{-w}) = \sum_{a \in A_w} \int_0^{x_a} t_a(u)du, \tag{12}$$

subject to

$$\sum_{r \in R^w} f_r^w = F^w, \tag{13}$$

$$f_r^w \geq 0, \quad \forall r \in R^w, \tag{14}$$

with definitional constraint

$$x_a = \sum_{w \in W} \sum_{r \in R^w} f_r^w \delta_{a,r}^w, \quad \forall r \in R^w, a \in A_w. \tag{15}$$

Let us formulate the problem (12)–(15) in a form of noncooperative network game with penalty functions: $\Omega\big(W, \{\mathbf{F}_w\}_{w \in W}, \{P_w\}_{w \in W}\big)$, where $\mathbf{F}_w = \{f^w | f_r^w \geq 0 \ \forall r \in R^w, \ \sum_{r \in R^w} f_r^w = F^w, \ \forall w \in W\}$, when

$$x_a = \sum_{w \in W} \sum_{r \in R^w} f_r^w \delta_{a,r}^w, \quad \forall r \in R^w, a \in A_w.$$

Nash equilibrium in game Ω is reached by such x^* that

$$P_w(x^*) \leq P_w\left(x^w, x^{-w^*}\right) \quad \forall w \in W,$$

where $x^{-w} = (\ldots, x^{w-1}, x^{w+1}, \ldots)$.

Let us offer the following theorem. Proof of the theorem is highly important from meaningful perspective.

Theorem 2. *User-equilibrium flow pattern (with continuous flows) is a pure strategy Nash equilibrium in game Ω.*

Proof. Assume

$$\sum_{a \in p} \int_0^{x_a} t_a(u)du > \sum_{a \in q} \int_0^{x_a} t_a(u)du,$$

Then

$$\sum_{a \in (p-q)} \int_0^{x_a} t_a(u)du - \sum_{a \in (q-p)} \int_0^{x_a} t_a(u)du = \delta > 0.$$

Due to continuity of $\int_0^{x_a} t_a(u)du$, there exist $\Delta f_1 > 0$, $\Delta f_2 > 0$, such that

$$\sum_{a \in (p-q)} \int_0^{x_a - \Delta f_1} t_a(u)du > \sum_{a \in (p-q)} \int_0^{x_a} t_a(u)du - \frac{\delta}{3}, \tag{16}$$

$$\sum_{a\in(q-p)}\int_0^{x_a+\Delta f_2} t_a(u)du > \sum_{a\in(q-p)}\int_0^{x_a} t_a(u)du + \frac{\delta}{3}. \qquad (17)$$

Let us substitude $\Delta f = \min\{\Delta f_1, \Delta f_2\}$ for Δf_1 and Δf_2 in (16) and (17) and the inequalities still hold. Thus, from (16) and (17) we obtain

$$\sum_{a\in(q-p)}\int_0^{x_a+\Delta f} t_a(u)du - \sum_{a\in(p-q)}\int_0^{x_a-\Delta f} t_a(u)du <$$
$$< \sum_{a\in(q-p)}\int_0^{x_a} t_a(u)du - \sum_{a\in(p-q)}\int_0^{x_a} t_a(u)du + \frac{2\delta}{3} = \frac{-\delta}{3} < 0. \qquad (18)$$

Let ΔP_w be the change in P_w resulting from a transfer of Δf from p to q. Since $\int_0^{x_a} t_a(u)du$ is increasing function

$$\Delta P_w < \sum_{a\in(q-p)}\int_0^{x_a+\Delta f} t_a(u)du\cdot\Delta f -$$
$$- \sum_{a\in(p-q)}\int_0^{x_a-\Delta f} t_a(u)du\cdot\Delta f < \frac{-\delta}{3}\Delta f < 0. \qquad (19)$$

Therefore, if no player w can lower his payoff P_w by an interpath flow transfer, then the network is user optimized. However, the equivalence to Nash equilibrium is not yet transparent.

Eventually, S. Devarajan made important conclusions that we cite further (Devarajan, 1981). Theorem 2 does not guarantee the equivalence of user-equilibrium and Nash equilibrium in game of OD-pairs Ω. Actually, user-optimization is a weak condition for Nash equilibrium. The problem is that the criterion for user-optimization, that a shift of Δf from path p to path q not decrease the cost to Δf, is a weak condition for Nash equilibrium. Pure strategy Nash equilibrium requires that the adoption of any new pure strategy by a player should not improve his payoff. This means we should be able to transfer any number of f (not necessarily equal) from as many paths to another set of paths (all connecting the same OD-pair w) and not register a drop in payoff P_w. To put it another way, given a pure strategy $(f_{p_i}, \ldots, f_{p_n})$, the user optimization criterion refers to changing only two of the f_{p_i}'s. For Nash equilibrium, we must test whether shifts which change up to all the f_{p_i}'s improve the payoff. Thus, every Nash equilibrium is a user optimal network but not vice versa. However if we can show that:

- the user optimal solution is unique, and
- Nash equilibrium always exists in this game.

then the two equilibrium concepts are, in fact, equivalent.

5. Game of Individual Users

This section is devoted to another alternative game-theoretic formulation of user-equilibrium assignment. All discussed here results are obtained by R.W. Rosenthal (Rosenthal, 1973).

Let us start from introductory citation of R.W. Rosenthal (Rosenthal, 1973). The individuals are assumed to be playing a game in which the pure strategies for each are the individuals' feasible paths. The payoffs (to be minimized) are the sums of the costs of the arcs used. Nash equilibria are sought. In this case these correspond to equilibria for the system. For general n-person games, however, one is not guaranteed that any Nash equilibria must exist; unless the individual strategy sets are extended to include all possible randomizations over the sets of pure strategies. (See Nash, 1951) (The cost of playing a randomized strategy is taken to be the expected cost over the relevant pure strategies.) spond to fractional solutions to the continuous-variables model . For this class of games, however, it is not necessary to introduce randomizations, since pure-strategy Nash equilibria always exist.

Denote x_a^k as the fraction of individual k's flow which passes through arc a, $k \in \{1, \ldots, |F|\}$, $a \in A$. If all individuals have chosen their routes, the total cost to an individual traversing route r is

$$P_k(r) = \sum_{a \in r} t_a(x_a). \tag{20}$$

Then we can formulate the following noncooperative network game: $\Upsilon(\mathbf{F}, R, \{P_k\}_{k \in \mathbf{F}})$, where $\mathbf{F} = \{1, \ldots, |F|\}$ and R is a set of all possible routes.

An equilibrium for the system is a set of feasible paths, one for each individual, such that no individual can decrease his total cost by switching unilaterally to some other feasible path. We shall assume in all that follows that at least one feasible route exists for each individual.

$$\min \sum_{a \in A} \sum_{u=0}^{x_a} t_a(u) \tag{21}$$

subject to

$$x_a = \sum_{k=1}^{|F|} x_a^k, \quad a \in A, \tag{22}$$

when

$$x_a^k = \begin{cases} 1, \text{if choosen route containing arc } a, \\ 0, \text{otherwise.} \end{cases} \tag{23}$$

The following theorem was proved (Rosenthal, 1973).

Theorem 3. *In game Υ, derived from network equilibrium models, pure-strategy Nash equilibria always exist. Furthermore, any solution to the problem (21)–(23) is a pure-strategy Nash equilibrium in game Υ.*

We can say, that theorem 3 gave the first justification of a high correlation between selfish routing and user equilibrium of Wardrop.

6. Game of Users' Groups

This section is devoted to relationships between user equilibrium of Wardrop, Nash equilibrium (competition between several groups of users) and system optimum of Wardrop. All discussed here results are obtained by A.Y. Krylatov *et al.* (Krylatov et al., 2016).

Let's start with the fact that the both Wardrop's principle are useful from practical perspective. However, the drawback is that they are not applicable when instead of atomic drivers, the behavior of groups of drivers (with common group interest) should be taken into consideration. Thus, by virtue of rapid guidance systems development the need for a new assignment principle is expected to increase significantly in the coming years. If the term "group of users (UG)" we understand as a set of all drivers following advices of the same guidance system, then a new principle could be formulated roughly as follows:

- Under competitive conditions the average journey time of each group of users is a minimum.

Note, the explicit mention of the competitive environments in this principle is important since without competitive behavior of companies creating guidance systems the second principle of Wardrop is sufficient for the purpose of appropriate modeling. Further we will associate this *competitive traffic assignment principle* with the *competitive traffic assignment problem.*

Consider the same transportation network presented by oriented graph $G = (N, A)$, set of OD pairs W and corresponding sets of routes R^w, $w \in W$. According to competitive traffic assignment principle, each group tries to assign its users among available routes from origin to destination in such a way, that their average travel time is minimum. Introduce following notation: $M = \{1, \ldots, m\}$ is the set of users' groups (UG); $F^{jw} > 0$ is the demand of UG j between w, $F^j = \sum_{w \in W} F^{jw}$; x_a^j is the flow of UG j throgh the arc $a \in A$, $x^j = (\ldots, x_a^j, \ldots)$, $x^{-j} = (x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^m)$ and $x_a = (x_a^1, \ldots, x_a^m)$; f_r^{jw} is the flow of UG j through $r \in R^w$; $f^{jw} = (f_1^{jw}, \ldots, f_{|R^w|}^{jw})^\top$ is the assignment of the flow F^{jw} through possible routes R^w; $f^j = (\ldots, f^{j,w}, \ldots)$ is the strategy of UG j (the assignment of the flows F^{jw} between all OD-pairs), and $f^{-j} = (f^1, \ldots, f^{j-1}, f^{j+1}, \ldots, f^m)$; $f = (f^1, ..., f^m)$ is the set of all strategies of all UGs.

Each UG tries to minimize the average travel time of its own users. Therefore, the following optimization programs could be formulated for all $j = \overline{1, m}$ Zakharov and Krylatov, 2016:

$$T_m^j \left(\mathbf{x}^{j^*}, x^{-j} \right) = \min_{x^j} T_m^j (x) = \min_{x^j} \sum_{a \in A} t_a(x_a) x_a^j, \tag{24}$$

subject to

$$\sum_{r \in R^w} f_r^{jw} = F^{jw} \quad \forall w \in W, \tag{25}$$

$$f_r^{jw} \geq 0 \quad \forall r \in R^w, w \in W, \tag{26}$$

with definitional constraints

$$x_a^j = \sum_{w \in W} \sum_{r \in R^w} f_r^{jw} \delta_{a,r}^w \quad \forall a \in A, \tag{27}$$

$$x_a = \sum_{j=1}^m x_a^j \quad \forall a \in A. \tag{28}$$

Note, that for each $j \in M$ the set x^{-j} is not fixed, but induced by the assignment decisions of other UGs. Therefore, we obtain competitive traffic assignment problem, that could be reformulated in a form of noncooperative network game with penalty functions z^j, $j = \overline{1,m}$: $\Gamma_m\big(M, \{F^j_m\}_{j\in M}, \{T^j_m\}_{j\in M}\big)$, where $F^j_m = \{f^{jw} | f^{jw}_k \geq 0 \ \forall k \in K_w, \ \sum_{k\in K_w} f^{jw}_k = F^{jw}, \ \forall w \in W\}$, when

$$x^j_a = \sum_{w\in W}\sum_{r\in R^w} f^{jw}_r \delta^{jw}_{a,r} \quad \text{and} \quad x_a = \sum_{j=1}^{m} x^j_a.$$

Consideration of competitive traffic assignment problem in a game theoretic form leads us to the Nash equilibrium search. Nash equilibrium in the game Γ_m is realized by strategies $\mathbf{x}^{ne}_m = (\mathbf{x}^{1^*}, \ldots, \mathbf{x}^{m*})$ such that

$$T^j_m\left(\mathbf{x}^{ne}_m\right) \leq T^j_m\left(x^j, \mathbf{x}^{-j^*}\right) \quad \forall j \in M. \tag{29}$$

Theorem 4. *The following inequalities hold*

$$T(\boldsymbol{x}^{so}) \leq T(\boldsymbol{x}^{ne}_m) \leq T(\boldsymbol{x}^{ue}). \tag{30}$$

Corollary 1. *If $t_a(x_a) \neq const$ then the following inequalities hold*

$$\begin{aligned} T(\boldsymbol{x}^{so}) = T(\boldsymbol{x}^{ne}_1) < T(\boldsymbol{x}^{ne}_2) < \ldots < T(\boldsymbol{x}^{ne}_m) < \\ < T(\boldsymbol{x}^{ne}_{m+1}) < \ldots < T(\boldsymbol{x}^{ne}_{|F|}) \leq T(\boldsymbol{x}^{ue}). \end{aligned} \tag{31}$$

The proof of Theorem 1 allow us to establish some rules that characterize the border conversion of competitive assignment (Nash equilibrium) into non-competitive assignment (user equilibrium and system optimum). Here they are:

- if $j = 1$ then Nash-equilibrium is converted into system optimum of Wardrop,
- if $F^j = 1$ for all $j = \overline{1,m}$ then Nash equilibrium in pure strategies is converted into *integer* user equilibrium of Wardrop,
- if $F^j = 1$ for all $j = \overline{1,m}$ then Nash equilibrium in mixed strategies is converted into user equilibrium of Wardrop,
- if $F^j \to 0$ for all $j = \overline{1,m}$ or equivalently $m \to \infty$ then Nash equilibrium is converted into user equilibrium of Wardrop.

Moreover, due to Corollary one could state three important conclusions:

1 Competitive in-vehicle route guidance systems decrease the average travel time in urban traffic area in comparison with an atomic vehicle guidance.
2 The less amount of competitive in-vehicle guidance systems, the less average travel time in urban traffic area.
3 Centralized guidance system guarantees the least travel time in urban traffic area.

Therefore, since in modern worldwide cities drivers chose routes by their own even competitive guidance systems could decrease the average travel time. Consequently, from this perspective, the development of intelligent vehicles could significantly improve traffic conditions in urban areas.

7. Conclusion

This paper was devoted to discussing approaches for modeling selfish and group routing. Surprisingly, first behavioral model for selfish routing appeared in 1950's, but no any methodological novations have appeared since then. Significant meaningful contribution was made by T. Roughgarden in 2000's. New relationships between user-equilibrium of Wardrop, Nash equilibrium (competiion between groups of users) and system optimum of Wardrop were established in 2010's. Thus, there is a clear opportunity for development of selfish and group routing models on the basis of game-theoretic approach.

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