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A Search Game with Incomplete Information on Detective Capability of Searcher*

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Abstract This paper deals with a so-called *search allocation games* (SAG), which a searcher distributes search resources, such as detection sensors and search time, into a search space to detect a target and the target moves to evade the detection. Although there have been many published papers on the SAG, they almost dealt with complete information games. In this paper, we consider private information of the searcher about the detection effectiveness of the search resource and discuss a two-person zero-sum incomplete information SAG with the detection probability of the target as payoff. We derive its Bayesian equilibrium to evaluate the value of the incomplete information.

Keywords: search theory, game theory, incomplete information.

1. Introduction

This paper deals with a so-called *search allocation games* (SAG) (Hohzaki, 2013a), which a searcher distributes search resources, such as detection sensors and search time, into a search space to detect a target and the target moves to evade the detection. Although there have been many published papers on the SAG, they almost dealt with complete information games. In this paper, we consider private information of the searcher about the detection effectiveness of the search resource and discuss a two-person zero-sum incomplete information SAG with the detection probability of the target as payoff. We derive its Bayesian equilibrium to evaluate the value of the incomplete information.

Morse and Kimball (1951) already discussed a search game in their well-known book as a control problem of submarine's passage in straits in 1951. In the history of search theory, researchers first solved optimal search problems for stationary targets and then for moving targets. Koopman (1957) had been studying optimal distribution problems of search resources to detect targets effectively. His problem became an origin of the research field of the so-called optimal resource allocation. In these one-sided search problems, they assumed that the searcher knew a probabilistic rule of the target's existence or movement. De Guenin (1961) and Kadane (1968) are researches for stationary target problem, and Pollock (1970), Iida (1972), Brown (1980) and Washburn (1983) are for moving target problems. Stone (1975) got together the theoretical results of these early researches in his book.

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After that, they regarded the existence distribution and movement of the target in search spaces, which had been assumed to be given by some probabilistic laws, as elements of the target's decision making and began to discuss search games between a searcher and a target. They handled target strategies of existence or movement in almost all models of the search game, but various types of searcher's strategies (Hohzaki, 2016). The search game with the distribution strategy of search resources as a searcher's strategy is referred to as *search allocation game* (SAG). Nakai (1988) and Iida et al. (1994) discussed SAGs with stationary targets. Iida et al. (1996) and Hohzaki and Iida (1998) dealt with moving target SAGs. Washburn and Hohzaki (2001) developed a SAG model of embedding practical factors, e.g. energy and geographical constraints, into the target movement and Hohzaki (2006) generalized the model. Dambreville and Le Cadre (2002) and Hohzaki (2008) took account of some practical features of search resource in their SAGs. Hohzaki (2007a) analyzed a SAG with occurrence of false contacts in a search space. Multi-stage, cooperative and nonzero-sum SAGs were studied by Hohzaki (2007b), Hohzaki (2009) and Hohzaki (2013b), respectively.

In the previous papers mentioned above, they modeled their SAGs as complete information games. Namely, the target and the searcher share all information about players as common knowledge. However, we can think of some cases that specific private information of players has serious effects on the outcome of games. At the beginning of the game, an initial position and an initial movement energy of the target must be known to the target. Hohzaki and Joo (2015) and Matsuo and Hohzaki (2017) first began to discuss incomplete information SAGs with target's private information about initial position and initial energy, respectively. On the searcher's side, just the searcher would properly know the effectiveness of search resources such as detection sensors because the searcher uses the resources to detect the target. The target tries to acquire the detection effectiveness of the sensors and gets its outlines although he cannot certainly know it. What extent he can know it to depends on its technological history, its patent, how open the technology is to the world, and so on. If it is a high technology, it would be deeply kept in secret and the target's estimation about its effectiveness would be rather uncertain.

In this paper, we model an incomplete information SAG with private information about search resource such as detection sensors. We derive its equilibrium to evaluate the value of the information. By some numerical examples, we do the evaluation in the concrete and analyze the effects of whether the target successes or fails to estimate the resource's effectiveness on the results of the game. The analysis gives us a lesson about R & D of sensing or search resource technology.

2. A Search Game with Searcher's Information about Sensors

The detection capability of searcher's sensors affects the results of search operations very much. Only the searcher naturally knows the capability. We consider a one-shot search game between a searcher and a target with the searcher's private information about the sensors' capability.

(A1) A search space consists of a discrete time space, denoted by $\boldsymbol{T} = \{1, \cdots, T\}$, and a discrete geographic space, denoted by $\boldsymbol{K} = \{1, \cdots, K\}$.

(A2) There are two players: a searcher and a target. Only the searcher knows the detection capability of searcher's sensors. The capability is indexed by sensor's types H. The target does not know the sensors' type the searcher currently uses but can guess it according to a distribution of the types $\{f(h), h \in H\}$, where $\sum_{h\in H} f(h) = 1$.

(A3) The target starts from its initial possible cells $S_0 \subseteq \boldsymbol{K}$ and moves in the space $\boldsymbol{K}$ as time elapses. His motion has some constraints; he can move from current cell i at time t to cells $N(i,t)$ at time $t+1$; he takes energy $\mu(i,j)$ to move from cell i to j; he possesses initial energy e_0 at time $t=1$; he can do nothing but staying at his current cell if he exhausts his energy. Let us denote a set of feasible target paths of satisfying all constraints above by Ω. If the target chooses a path $\omega \in \Omega$, he will be at cell $\omega(t) \in \boldsymbol{K}$ at time $t \in \boldsymbol{T}$.

(A4) To detect the target, the searcher distributes search resources using a specific type of sensors after an initial time τ, that is, the searchable time period is denoted by $\widehat{\boldsymbol{T}} = \{\tau, \tau+1, \cdots, T\}$. $\Phi(t)$ resources are available to the searcher at each time t. Let us denote a distribution plan of the type h of search resources by $\varphi_h = \{\varphi_h(i,t),\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$, where $\varphi_h(i,t)$ is the amount of resources to be scattered in cell i at time t. We call the searcher using the h-type resources or sensors the h-type searcher.

(A5) The distribution of x h-type resources in cell i gives the searcher the following detection probability

$$1 - \exp(-\alpha_i^h x) \tag{1}$$

only if the target is there. Parameter α_i^h indicates the detection effectiveness per unit h-type resource in cell i.

(A6) If the searcher detects the target, the searcher gets reward 1 and the target loses the same.

If we denote a residual energy of the target at the beginning of time t by $e(t)$, we can define the movement constraints in Assumption (A3) on a feasible target path $\omega \in \Omega$ starting from initial cell k, as follows:

(i) Initial cell condition: $\omega(1) \in S_0$
(ii) Movable cell conditions: $\omega(t+1) \in N(\omega(t), t),\ t = 1, \cdots, T-1$
(iii) Initial energy condition: $e(1) = e_0$
(iv) Preservation conditions of energy: $e(t+1) = e(t) - \mu(\omega(t), \omega(t+1)),\ t = 1, \cdots, T-1$
(v) Energy condition for a motion: $\mu(\omega(t), \omega(t+1)) \le e(t),\ t = 1, \cdots, T-1$

Ω is defined as a set of feasible paths of satisfying all conditions above.

As seen from Assumption (A6), the problem is a two-person zero-sum game with the detection probability as payoff. Before the derivation of the payoff function of the game, let us confirm the feasible region of the distribution plan φ_h of the h-type searcher. It is given by

$$\Psi_h \equiv \left\{ \varphi_h \;\middle|\; \sum_{i \in \boldsymbol{K}} \varphi_h(i,t) \le \Phi(t),\ \varphi_h(i,t) \ge 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \right\}, \tag{2}$$

from Assumption (A4). Assume that the h-type searcher takes a pure strategy φ_h and the target does a pure strategy ω. At time t, $\varphi_h(\omega(t), t)$ resources are effective to the target detection because the target is in cell $\omega(t)$. Therefore, the searcher has the detection probability of the target

$$R_h(\varphi_h, \omega) = 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t)\right)$$

as the payoff function from Equation (1).

We think of a target mixed strategy $\pi \equiv \{\pi(\omega),\ \omega \in \Omega\}$, where $\pi(\omega)$ is the probability of choosing a path ω. It feasible region is given by

$$\Pi \equiv \left\{ \{\pi(\omega)\} \,\middle|\, \sum_{\omega \in \Omega} \pi(\omega) = 1, \quad \pi(\omega) \geq 0, \omega \in \Omega \right\}.$$

By the h-type searcher's strategyφ_h and the target mixed strategy π, the searcher has the expected payoff as follows:

$$\begin{aligned} R_h(\varphi_h, \pi) &= \sum_{\omega \in \Omega} \pi(\omega) R_h(\varphi_h, \omega) \\ &= \sum_{\omega \in \Omega} \pi(\omega) \left\{ 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t)\right) \right\} \\ &= 1 - \sum_{\omega \in \Omega} \pi(\omega) \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t)\right). \end{aligned} \tag{3}$$

The h-type searcher wants to maximize the payoff. On the other hand, the target desires to minimize the following expected payoff:

$$\begin{aligned} R(\varphi, \pi) &= \sum_{h \in H} f(h) R_h(\varphi_h, \pi) \\ &= \sum_{\omega \in \Omega} \pi(\omega) \sum_{h \in H} f(h) \left\{ 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t)\right) \right\} \\ &= 1 - \sum_{\omega \in \Omega} \pi(\omega) \sum_{h \in H} f(h) \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t)\right), \end{aligned} \tag{4}$$

considering all types of searchers' strategies $\varphi \equiv \{\varphi_h, h \in H\}$ based on the probability distribution $\{f(h), h \in H\}$ of searcher's type. The expected payoff by a target pure strategy ω and all types of searchers' strategies φ, $R(\varphi, \omega)$, is given by

$$R(\varphi, \omega) \equiv \sum_{h \in H} f(h) R_h(\varphi_h, \omega) = \sum_{h \in H} f(h) \left\{ 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t)\right) \right\}.$$

At a glance, the payoff looks different between the searcher and the target. Let us discuss this search game and derive an equilibrium.

3. Equilibrium of the One-Shot Search Game

As seen from Equations (3) and (4), a set of optimal strategies φ_h of maximizing $R_h(\varphi_h, \pi)$ for every h maximizes $R(\varphi, \pi)$ in the aggregate. As a result, all types of searchers and the target play a two-person zero-sum game with the payoff $R(\varphi, \pi)$.

A max-min optimization of $R(\varphi, \pi)$ is all we have to do to obtain optimal searchers' strategies for all types. We can transform a minimization of $R(\varphi, \pi)$ with respect to π into

$$\min_{\pi} R(\varphi, \pi) = \min_{\pi} \sum_{\omega \in \Omega} \pi(\omega) \sum_{h \in H} f(h) R_h(\varphi_h, \omega)$$
$$= \min_{\omega \in \Omega} \sum_{h \in H} f(h) \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t) \right) \right\}. \tag{5}$$

We maximize the value minimized above with respect to φ to have the following formulation of the max-min optimization:

$$(P_S) \ \max_{\varphi, \nu} \ \nu$$
$$s.t. \ \sum_{h \in H} f(h) \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t) \right) \right\} \geq \nu, \ \omega \in \Omega, \tag{6}$$
$$\sum_{i \in \boldsymbol{K}} \varphi_h(i, t) \leq \Phi(t), \ t \in \widehat{\boldsymbol{T}}, h \in H, \quad \varphi_h(i, t) \geq 0, \ i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}, h \in H.$$

Considering that the feasible region of constraints on variables φ and ν is a convex set, the problem is a convex programming problem. We can numerically solve it to get optimal strategies φ^* of all types of searchers by a general commercial solver.

We note that the optimal h-type searcher's strategy φ^*_h maximizes $R_h(\varphi_h, \pi)$ for a given target strategy π and its maximized value is a real detection probability if the type h is realized to the searcher. Let us write down the maximization problem.

$$(P^h_S) \ D^*_h = \max_{\varphi_h} \ 1 - \sum_{\omega \in \Omega} \pi(\omega) \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t) \right)$$
$$s.t. \ \sum_{i \in \boldsymbol{K}} \varphi_h(i, t) \leq \Phi(t), \tag{7}$$
$$\varphi_h(i, t) \geq 0, \ i \in \boldsymbol{K}, \ t \in \widehat{\boldsymbol{T}}. \tag{8}$$

From now, we are going to derive an optimal target strategy. The optimal strategy π must be a best response to optimal searchers' strategies $\varphi^* = \{\varphi^*_h, h \in H\}$ of the problem (P_S) so that π minimizes $R(\varphi^*, \pi) = \sum_{\omega \in \Omega} \pi(\omega) \sum_{h \in H} f(h) R_h(\varphi^*_h, \omega)$. Conversely, φ^* must be a best response to the target strategy π and then be an optimal solution of the problem (P^h_S). Setting Lagrange multipliers $\lambda_h(t)$ and $\eta_h(i, t)$

corresponding to conditions (7) and (8), we define a Lagrange function by

$$L_h(\varphi_h; \lambda_h(t), \eta(i,t)) \equiv \sum_{\omega \in \Omega} \pi(\omega) \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi_h(\omega(t), t)\right) + \sum_{t \in \widehat{\boldsymbol{T}}} \lambda_h(t) \left(\sum_i \varphi_h(i,t) - \Phi(t)\right) - \sum_{i,t} \eta_h(i,t) \varphi_h(i,t)$$

and have the so-called Karush-Kuhn-Tucker (KKT) conditions as necessary and sufficient conditions of optimal solution of the problem (P^h_S), as follows:

$$\frac{\partial L_h}{\partial \varphi_h(i,t)} = -\alpha^h_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{t' \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t')} \varphi_h(\omega(t'), t')\right) + \lambda_h(t) - \eta_h(i,t) = 0, \tag{9}$$

$$\lambda_h(t) \left(\sum_{i \in K} \varphi_h(i,t) - \Phi(t)\right) = 0,\ t \in \widehat{\boldsymbol{T}}, \tag{10}$$

$$\eta_h(i,t)\varphi_h(i,t) = 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}, \tag{11}$$

$$\lambda_h(t) \geq 0,\ t \in \widehat{\boldsymbol{T}}, \tag{12}$$

$$\eta_h(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}, \tag{13}$$

$$\sum_{i \in K} \varphi_h(i,t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}}, \tag{14}$$

$$\varphi_h(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}, \tag{15}$$

where $\Omega_{it} \equiv \{\omega \in \Omega \mid \omega(t) = i\}$. We unify conditions (9)–(13) to have the following conditions for an optimal solution φ^*_h.

(i) If $\varphi^*_h(i,t) > 0$,

$$\alpha^h_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{t' \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t')} \varphi^*_h(\omega(t'), t')\right) = \lambda_h(t). \tag{16}$$

(ii) If $\varphi^*_h(i,t) = 0$,

$$\alpha^h_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{t' \in \widehat{\boldsymbol{T}}} \alpha^h_{\omega(t')} \varphi^*_h(\omega(t'), t')\right) \leq \lambda_h(t). \tag{17}$$

Because the optimal solution of the problem (P_S), $\varphi^* = \{\varphi^*_h\}$, satisfies conditions (14) and (15), we can regard these conditions (16) and (17) as the conditions of an optimal π^* such that φ^*_h becomes a best response to π. If the conditions hold for every type h, $\varphi^* = \{\varphi^*_h\}$ is aggregately the best response to π. From the discussion so far, we formulate the problem of giving an optimal target strategy π^* into the following:

$$(P_T)\ \min_{\pi,\lambda} \sum_{\omega\in\Omega} \pi(\omega) \sum_{h\in H} f(h) \left\{ 1 - \exp\left(- \sum_{t\in\widehat{\boldsymbol{T}}} \alpha^h_{\omega(t)} \varphi^*_h(\omega(t), t) \right) \right\}$$

$$s.t.\ \ \alpha^h_i \sum_{\omega\in\Omega_{it}} \pi(\omega) \exp\left(- \sum_{t'\in\widehat{\boldsymbol{T}}} \alpha^h_{\omega(t')} \varphi^*_h(\omega(t'), t') \right) = \lambda_h(t) \qquad (18)$$

$$for\ (i,t,h) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}} \times H\ of\ \varphi^*_h(i,t) > 0,$$

$$\alpha^h_i \sum_{\omega\in\Omega_{it}} \pi(\omega) \exp\left(- \sum_{t'\in\widehat{\boldsymbol{T}}} \alpha^h_{\omega(t')} \varphi^*_h(\omega(t'), t') \right) \le \lambda_h(t) \qquad (19)$$

$$for\ (i,t,h) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}} \times H\ of\ \varphi^*_h(i,t) = 0,$$

$$\sum_{\omega\in\Omega} \pi(\omega) = 1, \qquad (20)$$

$$\pi(\omega) \ge 0,\ \omega \in \Omega. \qquad (21)$$

4. Consistency of Equilibrium with Dynamic Game

The situation of the search game transits as time elapses. At each time on a time horizon, each player comes to acquire private or common information about the situation of the game. That is why we regard our one-shot game as a dynamic game changing on the time horizon. Here we confirm that the equilibrium of π^* and φ^*, which we derived in Section 3., is consistent with the dynamic game.

Assume that it is time t and the game has progressed to the time. Assume more that the target has not been detected and has residual energy e at cell i while moving along a pre-determined path ϖ. At the present time t, both players recognize the non-detection of the target but the selection of the path ϖ and the current state (i,e) are private information of the target. The realized type h of the searcher and the past distribution of search resources $\{\varphi^*_h(i,t'), i \in \boldsymbol{K}, t' = \tau, \cdots, t-1\}$ are a private information of the searcher. Each player can make a rational estimation on his opponent's strategy, φ^* or π^* from the problems (P_S) or (P_T).

For the game continuing after the current time t, rational decision making of the target and the searcher must have the following features;

(i) If the h-type searcher took a distribution plan $\varphi^{\tau}_{t-1} \equiv \{\varphi_h(i,t'), i \in \boldsymbol{K}, t' = \tau, \cdots, t-1\}$ in the past, he should revise his strategy after time t by rationally estimating the target strategy of path selection by

$$\pi'(\omega) = \frac{\pi^*(\omega) \exp\left(-\sum_{t'=\tau}^{t-1} \alpha^h_{\omega(t')} \varphi_h(\omega(t'), t') \right)}{\sum_{\omega'\in\Omega} \pi^*(\omega') \exp\left(-\sum_{t'=\tau}^{t-1} \alpha^h_{\omega'(t')} \varphi_h(\omega'(t'), t') \right)} \qquad (22)$$

in a Bayesian manner. The revised searcher's strategy coincides with the pre-planned strategy $\{\varphi^*_h(i,t'), t' = t, \cdots, T\}$. It would be proved later.

(ii) The target should revise the estimation on the probability of searcher's type h by

$$f'(h) = \frac{f(h) \exp\left(-\sum_{t'=\tau}^{t-1} \alpha_{\overline{\omega}(t')}^{h} \varphi_h^*(\overline{\omega}(t'), t')\right)}{\sum_{h' \in H} f(h') \exp\left(-\sum_{t'=\tau}^{t-1} \alpha_{\overline{\omega}(t')}^{h'} \varphi_{h'}^*(\overline{\omega}(t'), t')\right)} \tag{23}$$

in a Bayesian manner. Based on the estimation, the target should choose a path $\widetilde{\omega}$ starting from his current state (i,t,e) with probability

$$\pi(\widetilde{\omega}) = \frac{\sum_{\omega' \in \widetilde{\Omega}(i,t,e;\widetilde{\omega})} \pi^*(\omega')}{\sum_{\omega' \in \overline{\Omega}(i,t,e)} \pi^*(\omega')}. \tag{24}$$

$\overline{\Omega}(i,t,e)$ is a set of paths going through the state (i,t,e) and $\widetilde{\Omega}(i,t,e;\widetilde{\omega})$ is a set of paths starting from the state (i,t,e) and having the same route as path $\widetilde{\omega}$. The sets are defined by

$$\overline{\Omega}(i,t,e) \equiv \{\omega \in \Omega \mid \omega(t) = i,\ e(\omega,t) = e\}$$
$$\widetilde{\Omega}(i,t,e) \equiv \{\omega \in \overline{\Omega} \mid \omega(t') = \widetilde{\omega}(t'),\ t' = t+1, \cdots, T\},$$

where $e(\omega,t)$ is the amount of residual energy at time t on the path ω and is defined by $e(\omega,t) \equiv e_0 - \sum_{t'=1}^{t-1} \mu(\omega(t'), \omega(t'+1))$.

Let us prove the above claim is correct.

(i) A rational searcher's strategy: If the searcher recognizes no detection and rationally estimates the target path selection by Equation (22) at the time t, the searcher'd better make his strategy by solving the following problem:

$$(Q_S)\quad v_t^{*h} = \min_{\varphi_T^t} \sum_{\omega \in \Omega} \pi'(\omega) \exp\left(-\sum_{t'=t}^{T} \alpha_{\omega(t')}^{h} \varphi_h(\omega(t'), t')\right)$$
$$s.t. \sum_{i \in \boldsymbol{K}} \varphi_h(i,t') \le \Phi(t'), t' = t, \cdots, T,\ \varphi_h(i,t') \ge 0, i \in \boldsymbol{K},\ t' = t, \cdots, T,$$

where π' is fixed and φ_T^t is $\varphi_T^t \equiv \{\varphi_h(i,t'),\ i \in \boldsymbol{K}, t' = t, \cdots, T\}$. If the game continues up to the time t with no detection of the target, the searcher would take the strategy φ_T^{*t} derived from the problem (Q_S) and evaluate the minimum non-detection probability v_t^{*h} during $[t,T]$. At the initial time $t=1$, the searcher has to consider the non-detection probability during an early period $[\tau, t-1]$

$$\overline{P}_{t-1}^{h} \equiv \sum_{\omega' \in \Omega} \pi^*(\omega') \exp\left(-\sum_{t'=\tau}^{t-1} \alpha_{\omega'(t')}^{h} \varphi_h(\omega'(t'), t')\right)$$

and v_t^{*h} during the late period $[t,T]$, and make a plan $\varphi_{t-1}^{\tau} \equiv \{\varphi_h(i,t'), i \in \boldsymbol{K}, t' = \tau, \cdots, t-1\}$ during the period $[\tau, t-1]$ to minimize the total non-detection probability $\overline{P}_{t-1}^{h} v_t^{*h}$. As a result, we have the following problem for an optimal strategy

φ^{τ}_{t-1} from Equation (22):

$$\min_{\varphi^{\tau}_{t-1}} \overline{P}^h_{t-1} \min_{\varphi^t_T} \sum_{\omega \in \Omega} \frac{\pi^*(\omega) \exp\left(-\sum_{t'=\tau}^{t-1} \alpha^h_{\omega(t')} \varphi_h(\omega(t'), t')\right)}{\overline{P}^h_{t-1}}$$
$$\times \exp\left(-\sum_{t'=t}^{T} \alpha^h_{\omega(t')} \varphi_h(\omega(t'), t')\right)$$
$$= \min_{\varphi_h} \sum_{\omega \in \Omega} \pi^*(\omega) \exp\left(-\sum_{t'=\tau}^{T} \alpha^h_{\omega(t')} \varphi_h(\omega(t'), t')\right)$$

The problem is equal to the problem (P^h_S) formulated for the h-type searcher's optimal strategy.
(ii) A rational target strategy: Consider a target on a path $\overline{\omega}$ with a current state (i,t,e). Based on the estimation (23), the target has to make a path selection plan $\widetilde{\pi}$ of paths starting from the current state by solving the following problem:

$$\lambda^*_{(i,t,e)}(\overline{\omega}) = \max_{\widetilde{\pi}} \sum_{\widetilde{\omega} \in \overline{\Omega}(i,t,e)} \widetilde{\pi}(\widetilde{\omega}) \sum_{h \in H} \frac{f(h) \exp\left(-\sum_{t'=\tau}^{t-1} \alpha^h_{\overline{\omega}(t')} \varphi_h(\overline{\omega}(t'), t')\right)}{S(i,t,e;\overline{\omega})}$$
$$\times \exp\left(-\sum_{t'=t}^{T} \alpha^h_{\widetilde{\omega}(t')} \varphi_h(\widetilde{\omega}(t'), t')\right),$$

where $S(i,t,e;\overline{\omega}) \equiv \sum_{h' \in H} f(h') \exp\left(-\sum_{t'=\tau}^{t-1} \alpha^{h'}_{\overline{\omega}(t')} \varphi_{h'}(\overline{\omega}(t'), t')\right)$. If the target takes a path $\overline{\omega}$ at time $t = 1$, he reaches the state (i,t,e) undetected with probability $S(i,t,e;\overline{\omega})$. Therefore, the target would choose a path during $[1, t-1]$ rationally from the following optimization:

$$\max_{\overline{\pi}} \sum_{(i,e)} \sum_{\overline{\omega} \in \overline{\Omega}(i,t,e)} \overline{\pi}(\overline{\omega}) S(i,t,e;\overline{\omega}) \lambda^*_{(i,t,e)}(\overline{\omega})$$
$$= \max_{\overline{\pi}, \widetilde{\pi}} \sum_{(i,e)} \sum_{\overline{\omega} \in \overline{\Omega}(i,t,e)} \overline{\pi}(\overline{\omega}) \sum_{\widetilde{\omega} \in \overline{\Omega}(i,t,e)} \widetilde{\pi}(\widetilde{\omega}) \sum_{h \in H} f(h) \exp\left(-\sum_{t'=\tau}^{T} \alpha^h_{\omega(t')} \varphi_h(\omega(t'), t')\right)$$
$$= \max_{\pi} \sum_{\omega \in \Omega} \pi(\omega) \sum_{h \in H} f(h) \exp\left(-\sum_{t'=\tau}^{T} \alpha^h_{\omega(t')} \varphi_h(\omega(t'), t')\right).$$

The path ω is constructed by connecting $\overline{\omega}$ during $[1,t]$ and $\widetilde{\omega}$ after the time t and the probability of taking the paths $\overline{\omega}$ and $\widetilde{\omega}$ is expressed by $\pi(\omega) \equiv \overline{\pi}(\overline{\omega}) \cdot \widetilde{\pi}(\widetilde{\omega})$. As a result, we have the same objective function as Function (4) essentially. The target'd better choose his path ω with the probability $\pi^*(\omega) = \overline{\pi}(\overline{\omega}) \cdot \widetilde{\pi}(\widetilde{\omega})$ at $t = 1$ and obey the rule given by Equation (24) for an optimal path selection after time t.

We have made sure that in a state (i,t,e) on the process of the dynamic game, an equilibrium of the game starting from the state is consistent with an equilibrium of the one-shot game, φ^* and π^*, at $t = 1$. By Bayesian estimations of (22) and (23), the searcher keeps his optimal strategy φ^*_h at any time and the target revises his path selection plan by Equation (24).

5. Numerical Examples

Here we take some numerical examples to investigate the effects of information about the searcher's type on search games. We consider a search space consisting of a discrete time space $\boldsymbol{T} = \{1, \cdots, 6\}$ and 19 hexagonal cells $\boldsymbol{K} = \{1, \cdots, 19\}$, as shown in Figure 1. Cells 9 and 11 are obstacles to interrupt the passage of a target; the target cannot go through these cells.

The target is at Cell $S_0 = \{1\}$ at time $t = 1$ and intends to go to Cell 19 by $t = 6$. He can move to the cells located next to his current position and two-cell-distance cells every time. From Cell 1, for example, he can move to cells 1, 2, 3, 4, 5, 6, 8, 9 and 10 if energy allows him to do so. He consumes nothing to stay at the current cell but energy 1 to move to the next cells and 4 to the two-cell-distance cells, as parameter $\mu(i, j)$ is set. He has initial energy $e_0 = 8$ at time $t = 1$. The movement constraints allows the target to have 294 options of paths from Cell 1 to 19.

$\Phi(t) = 1$ search resource is available to the searcher every time $t = 2, \cdots, 6$ after $\tau = 2$. The detection effectiveness of the resource is different depending on cell. In Cells $8, 13, 14$ beneath obstacle cell 9 and in Cells $7, 12, 16$ beneath the other obstacle cell 11, the effectiveness is lower. The searcher has two types of resources, the effectiveness of which is $\alpha_i = 0.8$ in any cell except the cells specified above. In cell $j \in \{8, 13, 14, 7, 12, 16\}$, $h = 1$-type resource, say classic resource, has $\alpha_j = 0.2$ but $h = 2$-type resource, say improved resource, has $\alpha_j = 0.6$.

The target knows that the resource's type or the searcher's type is $h = 1$ or $h = 2$, but does not know which type of the resource the searcher uses in practice.

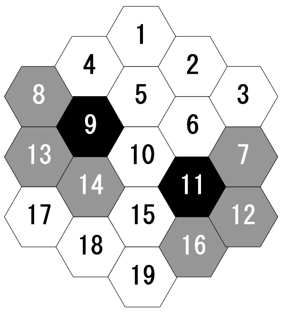

Figure 1. A search space

5.1. Sensitivity analysis of detection probability to $f(1)$

In Figure 2, a curve with circles indicates the expected detection probabilities or the optimized values of Problem (P_S) or (P_T) for $f(1)$ varying from 0 through 1. For the sake of comparison, we depict a curve with squares in the case of no private information, in which the target knows the searcher's type. We weight two game values for $h = 1$-type resource and $h = 2$-type resource with $f(1)$ and $f(2) = 1 - f(1)$, respectively, to calculate an expected detection probability in the case of no private information. Two curves coincide at the points of $f(1) = 0$ and $f(1) = 1$ because both points mean the target's recognition of the resource type. The difference between both curves is the value of the searcher's private information about the resource's type. As the searcher has some advantage of knowing his own type, the curve with private information always lies higher than that with no private information. The value of information becomes largest around point $f(1) = 0.5$, where the uncertainty of the searcher's type is largest to the target.

Table 1 gives us the detailed results of the numerical example above. From the left, the following numbers are listed in each column: $f(1)$, the value of the game with private information, the value of the game with no private information, the value of information explained above, and $R_h(\varphi_h^*, \pi^*)$ for $h = 1, 2$, calculated by Equation (3). Number in the last column is $R(\varphi^*, \pi^*) - R_1(\varphi_1^*, \pi^*)$ in the case of $f(1) < 0.5$ and $R(\varphi^*, \pi^*) - R_2(\varphi_2^*, \pi^*)$ in the other case of $f(1) > 0.5$. The number indicates how much the target's wrong estimation about the resource's type affects the detection probability. The number is transformed into

$$R(\varphi^*, \pi^*) - R_h(\varphi_h^*, \pi^*) = \begin{cases} (1 - f(1))(R_2(\varphi_2^*, \pi^*) - R_1(\varphi_1^*, \pi^*)), \mathit{if}\ h = 1 \\ (1 - f(2))(R_1(\varphi_1^*, \pi^*) - R_2(\varphi_2^*, \pi^*)), \mathit{if}\ h = 2 \end{cases}$$

and proportional to $1 - f(h)$ and $R_1(\varphi_1^*, \pi^*) - R_2(\varphi_2^*, \pi^*)$. That is why the number tends to be larger as the target estimates the resource's type more wrong. As seen from Table 1, the wrong estimation affects the detection probability larger in the case of $f(1) > 0.5$ than in the case of $f(1) < 0.5$. Let us analyze the difference between the two cases by taking $f(1) = 0.1$ and $f(1) = 0.9$ as their representatives.

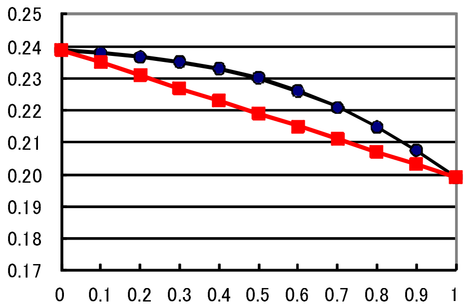

Figure 2. Game values for $f(1)$

Table 1: Sensitivity analysis of the game to $f(1)$

$f(1)$	$R(\varphi^*, \pi^*)$	*past model*	*value of info.*	$R_1(\varphi_1^*, \pi^*)$	$R_2(\varphi_1^*, \pi^*)$	$R - R_h$
0	0.2390	0.2390	0.0000	0.1379	0.2390	0.1011
0.1	0.2380	0.2351	0.0030	0.2280	0.2391	0.0102
0.2	0.2368	0.2311	0.0057	0.2256	0.2396	0.0112
0.3	0.2352	0.2271	0.0081	0.2229	0.2405	0.0123
0.4	0.2331	0.2231	0.0100	0.2184	0.2430	0.0147
0.5	0.2303	0.2192	0.0111	0.2133	0.2472	*N/A*
0.6	0.2261	0.2152	0.0110	0.2072	0.2546	−0.0285
0.7	0.2210	0.2112	0.0098	0.2042	0.2601	−0.0391
0.8	0.2149	0.2072	0.0077	0.2018	0.2673	−0.0524
0.9	0.2077	0.2032	0.0044	0.1997	0.2797	−0.0720
1	0.1993	0.1993	0.0000	0.1993	0.1810	0.0183

5.2. Effects of wrong estimation about the resource's type

Let us check an optimal movement strategy of the target first. The $h = 1$-type resources are much less effective for the searcher to detect the target moving on paths running on the right and left sides of the search space but are comparatively effective to detect the target moving in the central area between two obstacle cells. The $h = 2$-type resource is improved from the $h = 1$-type one in terms of its detection effectiveness in the side areas on the left side of Cell 9 and the right side of Cell 11. The target with an estimation of the $h = 1$-type resource would more likely take paths running in the side areas than the target with the other estimation of the $h = 2$-type resource. Practically, we can confirm the tendency by checking that the probabilities of target's taking paths going in the side areas and paths passing through the central area are 0.395 and 0.605, respectively, for $f(1) = 0.9$, and 0.344 and 0.656 for $f(1) = 0.1$. We must note that 91 percentages of 294 all paths go through the central area, though. Additionally, the target should use a mixed strategy of taking many paths running in various areas to make the searcher difficult to have a good estimate about the target path. Therefore, the probabilities of taking side paths are not extremely biased even for $f(1) = 0.9$.

Each type of searcher makes a rational distribution plan of resources, φ_h^*, considering the features of the target strategy explained above. In the case of $f(1) = 0.9$, in which the target more likely believes the searcher's possession of the $h = 1$-type resource, the $h = 2$-type searcher with the $h = 2$-type resource puts a lot of the improved resources in side areas to increase the detection probability. In the case of $f(1) = 0.1$, in which the target has more belief of the searcher's usage of the $h = 2$-type resource, the target does not bias the path selection in favor of side paths but evenly takes the paths going through the central area and the side areas. Under the situation, the $h = 1$-type searcher can keep the detection probability being not so worse even if he has to use less effective $h = 1$-type resources in the side areas, as seen from the cases of $f(1) = 0 - 0.5$ in Table 1.

As discussed so far, the effect of the wrong estimation about the resource's type is getting bigger as $f(1)$ is increasing. But we regard the cases of $f(1) = 0$ and $f(1) = 1$ as exceptions because we would properly think that the target has a confidence in the resource's type in the both cases.

From the analyses above, the searcher has a lesson about the usage of new and old sensors. It would say that developing of new technological sensors and installing them are more effective in search operations when the searcher pretends to possess no advanced sensor to his opponents. Conversely, to the target with a belief of the searcher's possession of advanced sensors, even the usage of old-fashion sensors does not make the detection probability so worse. The lesson would be valid just in the cases set in this section because of case-dependency usually. We just demonstrate the possibility of these analyses.

6. Conclusion

In this paper, we constructed a general model of incomplete information SAG with searcher's private information about search resource such as detection sensors and proposed a method to derive its equilibrium to evaluate the value of the information. By some numerical examples, we concretely evaluated the value of the information about the high-tech and the old-fashion sensors and analyzed the effects of target's guess at the sensor's detection effectiveness on the detection probability of the target in a search operation. From the analysis, we demonstrated the discussion about the value of the research and development of sensing technology. We handled the incomplete information about the detection effectiveness of the search resource in this paper but we could extend our procedure to other properties or characteristics of the resource uncertain to the target.

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