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On the Conditions on the Integral Payoff Function in the Games with Random Duration*

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Abstract In this paper we consider the problem of the existence of the integral payoff in the differential games with random duration when the random time is defined on the infinite time interval. We present an example of a game with random duration, a game-theoretic model of the development of non-renewable resources.
Keywords: differential games, random duration, environment, pollution control.

1. Introduction

When solving various economical problems it is advantageous to use game-theoretic models with random time horizon (Petrosjan and Shevkoplyas, 2003). If the random time instant is defined on the infinite interval, the payoff functional turns out to be an improper integral. Thus, the problem of convergency of such integrals arises. In the classical differential games with the infinite time horizon it is common to add a discounting factor to the model (Dockner et al., 2000). In the work (Aseev and Kryazhimskiy, 2007) the problem of convergency of the integral payoff in deterministic games on the infinite interval was solved. In this paper we generalize that result and present sufficient conditions for convergency of the expected payoff functional for games with random duration (Proposition 1) and random initial time (Proposition 2).

The paper is structured as follows. Section 1 contains a general formulation of the differential games with random time horizon. In section 2 we briefly overview the problem of transformation of the expected payoff function. In section 3 we discuss the problem of convergency and describe the ways to solve it. The example of the game of extraction of non-renewable resources is presented in section 4. In section 5 we generalize the results to the class of games with random initial time.

2. Game formulation

Consider a differential game $\Gamma(x_0, t_0, T)$ with n players. The state equations have the form:

$$\dot{x} = g(x, u_1, ..., u_n), \quad x \in R^n, \quad u_i \in U \subset compR^l, \quad x(t_0) = x_0. \tag{1}$$

The game starts from initial state x_0 at the time t_0. We assume that the duration of the game is the random variable T with known probability distribution function $F(t)$, $t \in [t_0, \infty)$ (Petrosjan and Murzov, 1966; Petrosjan and Shevkoplyas, 2003).

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We assume that for all admissible controls of players there exists a piece-wise differentiable and extensible on $[t_0, \infty)$ solution of (1).

Let $h_i(x(\tau), u_1, ..., u_n)$ be the instantaneous payoff function of player i at the time $\tau, \tau \in [t_0, \infty)$ or briefly $h_i(\tau)$. The instantaneous payoff function of each player is assumed to be a continuous function of its arguments. The expected integral payoff of the player i can be written as

$$K_i(x_0, t_0, u_1, ..., u_n) = \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(\tau) d\tau dF(t), \quad i = 1, ..., n. \tag{2}$$

3. Transformation of integral functional

The transformation of integral functional in the form of double integral (2) to the standard for dynamic programming form is important for further study of the game. For the case of nonnegative instantaneous payoff function $h_i(\tau)$ the result was presented in the work (Shevkoplyas, 2014).

Theorem 1. *Let the instantaneous payoff function $h_i(\tau)$, for each player $i = 1, \dots, n$ be nonnegative for all $t \in [t_0, \infty)$ and measurable function of t. Then, expected payoff of player i (2) can be written as follows:*

$$K_i(x_0, t_0, u_1, \dots, u_n) = \int_{0}^{\infty} \int_{0}^{t} h_i(\tau) d\tau \, dF(t) = \int_{0}^{\infty} (1 - F(t)) h_i(\tau) d\tau. \tag{3}$$

This result was also used in (Boukas et al., 1990; Marin-Solano and Shevkoplyas, 2011). In the general case the result was obtained in the paper (Kostyunin and Shevkoplyas, 2011).

Theorem 2. *The expected payoff (2) can be written in the form (3), if the following conditions hold*

1.
$$\lim_{T \to \infty} (1 - F(T)) \int_{t_0}^{T} h_i(t) dt = 0. \tag{4}$$

2. *The following integrals exist in the sense of improper Riemann integrals:*
$$\int_{t_0}^{\infty} \left| \int_{t_0}^{t} h_i(\tau) d\tau \right| dF(t) < +\infty, \quad i = 1, \dots, n. \tag{5}$$

Further on we assume that conditions (4) and (5) hold and the transformation of double integral (2) to (3) takes place.

4. Convergency of expected payoff

Since the expected payoff function (3) of player i is an improper integral, it is necessary to ensure its existence. Remark that the multiplier $1 - F(t)$ can be written as follows:

$$1 - F(t) = e^{-\int_{t_0}^{t} \lambda(s) ds}, \quad \forall t \in [t_0; \infty), \tag{6}$$

where $\lambda(s) = \frac{f(s)}{1-F(s)}$ is the hazard function.

Thus, the expected payoff takes form

$$K_i(x_0, t_0, u_1, ..., u_n) = \int_0^\infty e^{-\int_{t_0}^\tau \lambda(s)ds} h_i(\tau)d\tau, \quad i = 1, ..., n. \tag{7}$$

The multiplier (6) can be treated as a discounting multiplier. However, in a general problem statement (Marin-Solano and Shevkoplyas, 2011) another discount component $e^{-\rho(t_0,t)}$ can be added and the expected payoff takes the form

$$K_i(x_0, t_0, u_1, ..., u_n) = \int_0^\infty e^{-\rho(t_0,\tau)-\int\limits_{t_0}^{\tau} \lambda(s)ds} h_i(\tau)d\tau, \quad i = 1, ..., n. \tag{8}$$

The presence of discount multiplier does not guarantee the convergence of the integral (8). The following proposition gives sufficient conditions for convergency of improper integral (8).

Proposition 1. *The following inequalities should hold for all admissible pair* (x, u)

$$e^{-\rho(t_0,t)-\int_{t_0}^t \lambda(s)ds} \max_{u(t)\in U} |h(x(t), u)| \le \mu(t), \quad t \ge t_0,$$

$$\int_t^\infty e^{-\rho(t_0,\tau)-\int_{t_0}^\tau \lambda(s)ds} |h(x(\tau), u(\tau))| d\tau \le \omega(t), \quad t \ge t_0,$$

where $\mu(t)$, $\omega(t)$ *– some positive functions of argument* t *such as* $\lim\limits_{t\to\infty} \mu(t) = +0$ *and* $\lim\limits_{t\to\infty} \omega(t) = +0$.

This proposition is the generalization of the work (Aseev and Kryazhimskiy, 2007) for the case of random duration of the game. In particular, the result by Aseev, Kryazhimskiy is recovered in the case of exponential distribution of the random variable T ($\int_{t_0}^\tau \lambda(s)ds = \lambda(\tau - t_0)$).

5. Example

As an example, consider a game-theoretic model of emissions management (see (Breton et al., 2005)). There are n players in the game, each has industrial production on its territory. It is assumed that the production volume is directly proportional to emissions u_i. Thus, the strategy of the player is the choice of the volume of harmful emissions $u_i \in [0, b_i]$. The solution is sought in the class of program strategies $u_i(t)$.

The income of the player i at time t is determined by the formula:

$$r_i(u_i(t)) = u_i(t)(b_i - 1/2u_i(t)). \tag{9}$$

The dynamic in total pollution x defined by the equation

$$\dot{x} = \sum_{i=1}^n u_i(t), \quad x(t_0) = x_0. \tag{10}$$

Each player bears the costs associated with removing contaminants. Instant payoff (utility) of the player i is equal to $r_i(u_i(t)) - d_i x(t), d_i > 0$.

Without loss of generality, we assume that the start of the game is $t_0 = 0$. In contrast to the model (Breton et al., 2005) we assume that the game has a random terminal time T, where T — random variable with the distribution function $F(t) = 1 - e^{-t^2}, t \geq 0$, which corresponds to the Weibull distribution with scale parameter $\lambda = 1$ and the parameter $\delta = 2$. The value $\delta = 2$ corresponds to the increase function of the failure rate $\lambda(t) = \frac{f(t)}{1-F(t)}$, which can be interpreted as depreciation of equipment in the workplace.

The expected payoff of player i for the considered model has the form

$$K_i(0, x_0, u_1, \ldots, u_n) = \int_0^\infty \int_0^t (r_i(u_i(\tau)) - d_i x(\tau)) d\tau \ 2te^{-t^2} dt. \tag{11}$$

It was shown in (Kostyunin and Shevkoplyas, 2011) that for this game the conditions of Theorem 2 are hold and (11) can be transformed to the form

$$K_i(0, x_0, u_1, \ldots, u_n) = \int_0^\infty (r_i(u_i(t)) - d_i x(t)) \ e^{-t^2} dt. \tag{12}$$

Show that for (12) all conditions of Proposition 1 hold and therefore the integral is convergent.

We will use the following estimations: $x(\tau) \leq x_0 + \sum_{i=1}^n b_i \tau = x_0 + B\tau$, and also $r_i(u_i(\tau)) \leq \frac{b_i^2}{2}$, where $B = \sum_{i=1}^n b_i$.

We should find corresponding positive functions $\mu(t)$, $\omega(t)$ such that $\lim_{t\to\infty} \mu(t) = +0$ and $\lim_{t\to\infty} \omega(t) = +0$.

Consider an expression from first condition and evaluate it

$$e^{-t^2} \max_{u(t)\in U} |(r_i(u_i(t)) - d_i x(t))| \leq e^{-t^2} \max_{u(t)\in U} (|(r_i(u_i(t))| + |d_i x(t))|) \leq$$

$$\leq e^{-t^2} \left(\frac{b_i^2}{2} + (x_0 + Bt) \right).$$

Denote

$$\mu(t) = e^{-t^2} \left(\frac{b_i^2}{2} + (x_0 + Bt) \right).$$

We have $x_0 \geq 0$, hence we can conclude that $\mu(t)$ is a positive function. Since the linear functions grow asymptotically slower then exponential functions,

$$\lim_{t\to\infty} \mu(t) = +0.$$

The second condition of the proposition also holds. Indeed,

$$\int_t^\infty e^{-\tau^2} |(r_i(u_i(\tau)) - d_i x(\tau))| d\tau \leq \int_t^\infty e^{-\tau^2} (|(r_i(u_i(\tau))| + |d_i x(\tau))|) d\tau \leq$$

$$\leq \int_t^\infty e^{-\tau^2}(\frac{b_i^2}{2} + d_i(x_0 + B\tau))d\tau =$$

$$= \frac{1}{2}\left(d_i B e^{-t^2} + \sqrt{\pi}\ erfc(t)(d_i x_0 + \frac{b_i^2}{2})\right), \quad (13)$$

where $erfc(t)$ is the Gauss error function.

We can also rewrite (13) as follows

$$\frac{1}{2}(d_i B e^{-t^2} + \sqrt{\pi}\ erfc(t)(d_i x_0 + \frac{b_i^2}{2})) = \frac{1}{2}(d_i B e^{-t^2} + 2\sqrt{\pi}\ \Phi(\sqrt{2}t)(d_i x_0 + \frac{b_i^2}{2})),$$

where $\Phi(\sqrt{2}t)$ is the standard normal cumulative distribution function.

Therefore, we can denote

$$\omega(t) = \frac{1}{2}(d_i B e^{-t^2} + 2\sqrt{\pi}\ \Phi(\sqrt{2}t)(d_i x_0 + \frac{b_i^2}{2})).$$

Obviously, $\omega(t)$ is positive and

$$\lim_{t\to\infty} \frac{1}{2}(d_i B e^{-t^2} + 2\sqrt{\pi}\ \Phi(\sqrt{2}t)(d_i x_0 + \frac{b_i^2}{2})) = +0.$$

Thus, for this game formulation we show existence of functions $\mu(t)$ and $\omega(t)$ satisfying the relevant conditions.

6. Convergency in the games with random initial time

We can generalize the result for the class of game-theoretical models with random initial time (Gromova and Lopez-Barrientos, 2015). Consider a differential game $\hat{\Gamma}(x_0, t_0)$ with n players. The state equations have the same form as (1).

Suppose that the game start at the moment t_0 which is the random variable with known probability distribution function $\hat{F}(t)$, $t \in [t_0, \infty)$. Also assume that the game has infinite duration. Define the instantaneous payoff function in the same way as for the game $\Gamma(x_0, t_0, T)$.

Thus, the expected integral payoff of the player i can be written as the following Lebesgue-Stieltjes integral:

$$K_i(x_0, t_0, u_1, ..., u_n) = \int_{t_0}^\infty \int_t^\infty h_i(\tau) d\tau d\hat{F}(t), \quad i = 1, ..., n. \quad (14)$$

It is rather easy to show that (14) can be transformed to the standard for dynamic programming form (see (Gromova and Lopez-Barrientos, 2015)):

$$K_i(x_0, t_0, u_1, ..., u_n) = \int_{t_0}^\infty h_i(\tau)\hat{F}(\tau) d\tau, \quad i = 1, ..., n. \quad (15)$$

Since (15) is an improper integral, it is necessary to ensure its existence. The multiplier $\hat{F}(t)$ can be interpreted as a discounting component. The following propositions gives conditions for convergency of the integral.

Proposition 2. *The following inequalities should hold for all admissible pairs* (x, u)

$$\hat{F}(t) \max_{u(t)\in U} |h(x(t), u)| \leq \mu(t), \quad t \geq t_0,$$

$$\int_t^\infty \hat{F}(\tau)|h(x(\tau), u(\tau))|d\tau \leq \omega(t), \quad t \geq t_0,$$

where $\mu(t)$, $\omega(t)$ *– some positive functions of argument* t *such as* $\lim_{t\to\infty} \mu(t) = +0$ *and* $\lim_{t\to\infty} \omega(t) = +0$.

In the following, we plan to check conditions of Propositions 1 and 2 for different classes of distribution functions $F(t)$ and $\hat{F}(t)$.

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