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Static Game Theoretic Models of Coordination of Private and Public Interests in Economic Systems*

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Abstract A problem of inefficiency of equilibria (system compatibility) in static game theoretic models of resource allocation is investigated. It is shown that the system compatibility in such models is possible if and only if all agents are individualists or collectivists. Administrative and economic control mechanisms providing the system compatibility are analyzed.

Keywords: coordination of interests, hierarchical games, public goods economy, resource allocation.

1. Introduction

A problem of coordination of interests plays a key role in the investigation of social-economic systems based on mathematical modeling. The main research directions are theory of active systems (Mechanism design and management, 2013; Novikov, 2013), information theory of hierarchical systems (Gorelik and Kononenko, 1982), theory of contracts (Laffont and Martimort, 2002), mechanism design (Algorithmic Game Theory, 2007). An important role belongs to the notion of price of anarchy which characterizes a degree of coordination of interests of the active agents (Papadimitriou, 2001). The paper is dedicated to static game theoretic models of coordination of private and public interests (CPPI-models) in resource allocation in economic systems. In a seminal paper by Germeier and Vatel (1975) the models in which payoff functions of all players consist of two parts: public one (the same for all players) and private one, were analyzed. It is shown that if the payoff function is a convolution by minimum then with natural propositions a Pareto-optimal Nash equilibrium exists in the game (the price of anarchy is equal to one), and an ideal coordination of interests is possible. The research was developed, for example, by Kukushkin (1994). A powerful stream of literature in this domain belongs to the public goods economy which studies an optimal allocation of the resources of active agents between a production of a social good and their private activity (Bergstrom et al., 1986; Boadway et al., 1989a,b; Warr, 1983). Among recent papers are, for example, (Christodoulou et al., 2015) which is concerned with a mechanism of proportional allocation of divisible resources, and (Kahana and Klunover, 2016) in which the conditions of optimal allocation of the resources between leisure and labor are received in the case when individuals have the same utility function but different abilities and non-labor incomes. It should be

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noticed that a complete coordination of private and public interests is attained extremely rare, and special control mechanisms are designed to provide it. In a seminal paper by Burkov and Opoitsev (1974) was proposed an idea of optimal synthesis of a game of active agents in which the Nash equilibrium is profitable to the whole active system (the same idea is developed in mechanism design). The authors' papers (Gorbaneva and Ougolnitsky, 2013, 2015) deal with analysis of the system compatibility in resource allocation and building of the respective control mechanisms. A monograph by Gorbaneva et al. (2016) describes modeling of corruption in the hierarchical control systems. The corruption is treated as an additional feedback on the bribe and a specific way of coordination of interests. The rest of the paper is organized as follows. The problem setup is given in the Section 2. A possibility of the system compatibility in the model of coordination of private and public interests is studied in the Section 3. Administrative and economic control mechanisms which provide system compatibility or at least permit to approach to it are introduced and analyzed in the Section 4. The Section 5 concludes.

2. The problem setup

Denote by $N = \{0, 1, 2, \dots, n\}$ a set of elements of an economic system, where $\{0\}$ is the leader (Principal), and $M = \{1, 2, \dots, n\}$ is a set of followers (agents). The following types of games are considered:
(a) a game in normal form of equal players:
$\Gamma_{eq} =< M, \mathbf{U}, \mathbf{J}(\mathbf{u}) >=< \{1, 2, \dots, n\}, (U_1, U_2, \dots, U_n), (g_1, g_2, \dots, g_n) >$.
(b) a hierarchical game:
$\Gamma_{hi} =< N, \mathbf{U}, \mathbf{J}(\mathbf{u}) >=< \{0, 1, 2, \dots, n\}, (U_0, U_1, U_2, \dots, U_n), (g_0, g_1, g_2, \dots, g_n) >$.
Here U_i is a set of strategies of the i-th player, u_i is a strategy of the i-th player ($u_i \in U_i$), and g_i is a payoff function of the i-th player.
The games are considered in the context of resource allocation between public and private interests (objectives). The game theoretic models are based on the approaches by information theory of hierarchical systems (Germeier and Vatel (1975); Kukushkin (1994)) and public goods economy (Bergstrom et al., 1986; Boadway et al., 1989a,b; Warr, 1983; Christodoulou et al., 2015; Kahana and Klunover, 2016).
In the game Γ_{eq} each player has an amount of resources r_i which he allocates between public and private interests (production of a public good and his private economic activity, respectively). Strategies of players u_i are amounts of resources assigned to the production of a public good (public interests). The rest $r_i - u_i$ finances his private activity. In this case $U_i = [0, r_i]$.
In the hierarchical game Γ_{hi} it is assumed that the leader has an amount of resources r which she allocates between a lower control level and her private activity. In turn, the lower control level shares his part between his followers and his private interests. Thus, in the hierarchical game a strategy of the i-th player is a share u_i from the amount of resources $ru_1u_2u_{i-1}$ assigned to the public objectives. In this case $U_i = [0, 1]$.
In both setups it is supposed that the public income is divided among the agents completely. The payoff function of each agent consists of two summands reflecting his private income and his share in the public income, respectively:

$$g_i = p_i(r_i - u_i) + s_i c(\bar{u}), \tag{1}$$

where $\bar{u} = (u_1, \ldots, u_n)$ is a vector of resources assigned by the players to the production of a public good (public income);
$c(\bar{u})$ is a function of the public income;
$p_i(r_i - u_i)$ is a function of the private income of the i-th player;
s_i is a share of the i-th player in the public income, $i \in M$.
Thus, in the models of coordination of private and public interests (CPPI-models) the games described above are specified as follows:
(A) a game in normal form of the equal players:

$$\Gamma_{eq} =< M, \mathbf{U}, \mathbf{J}(\mathbf{u}) >=< \{1, 2, \ldots, n\}, U_i = [0, r_i], g_i = p_i(r_i - u_i) + s_i c(\bar{u}) > . \quad (2)$$

(B) a hierarchical game:

$$\Gamma_{hi} =< N, \mathbf{U}, \mathbf{J}(\mathbf{u}) >=< \{0, 1, 2, \ldots, n\}, U_i = [0, 1], g_i = p_i(r_i - u_i) + s_i c(\bar{u}) > .$$

For building the hierarchical game on the base of a game in normal form the set of players M is added by a specific player 0 (leader, Principal) which represents the interests of the whole system. The set of strategies is added by the Principal' s vector strategy $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ which is a set of control impacts on other players depending on the type of the used control mechanism. Also, the vector of payoff functions is added by the Principal's payoff which is a function of social welfare equal to the sum of the payoffs of all agents given the condition $\sum_{i \in M} s_i = 1$.

$$g_0 = \sum_{i \in M} p_i(r_i - u_i) + c(\bar{u}). \quad (3)$$

To provide the system compatibility the following control mechanisms may be used:
(a) an administrative mechanism (compulsion) when the Principal constraints the sets of strategies of the agents, namely, she fixes the amounts q (scalar or vector ones) such that an agent cannot assign greater or less resources to the public objectives:

$$\pi_a = \{k_i = q_i = (\bar{q}_i, \underline{q_i}) | 0 \le \bar{q}_i, \underline{q_i} \le r_i, \underline{q_i} \le u_i \le \bar{q}_i\}.$$

In this case we receive $u_i \in U_i(q) \subset U_i$, where $\bar{q} = (q_1, q_2, \ldots, q_n)$ is a matrix of the dimension $n * 2$, $\bar{u} = (u_1, u_2, \ldots, u_n)$, and the social welfare function is

$$g_0 = \sum_{i \in M} p_i(r_i - u_i) + c(\bar{u}) - C(q). \quad (4)$$

where $C(q)$ is a function of administrative control costs;
(b) an economic mechanism (impulsion) when the Principal impacts the agents' payoff functions, namely, she sets the shares s_i of their participation in the public income:

$$\pi_e = \{k_i = s_i | 0 \le s_i \le 1, \sum_{i \in M} s_i = 1\}. \quad (5)$$

Each of the two mechanisms can be applied with a feedback or without it. If the feedback is present then we receive a Germeier game with a mechanism

$$\pi_G = \{k_i = k_i(\bar{u})\}. \quad (6)$$

otherwise a Stackelberg game arises with a mechanism

$$\pi_{St} = \{k_i = const\}. \quad (7)$$

The Germeier games may be accompanied by a corruption mechanism

$$\pi_b = \{k_i = k_i(\bar{u}, \bar{b})\}, \bar{b} = (b_1, b_2, \ldots, b_n). \tag{8}$$

where $b_i \in [0; 1]$ is a share of the bribe given to a bribe-taker by the agent. It is supposed here that a Principal - agents hierarchy is added by another element (or elements) , a supervisor. The Principal is not corrupted but the real control from her name is made by the supervisor which can weaken the Principal's requirements in exchange to the bribe from an agent. The models of corruption are considered in details in (Gorbaneva et al. (2016)).

In general case the considered game has the form

$$\Gamma_{attr} =< N, \bar{k} \in K, \{V_i\}_{i=1}^n, \bar{J}, \Pi > . \tag{9}$$

where N is a set of players; *attr* denotes the game's type (in normal form of hierarchical); $\bar{k}$ is a vector of the Principal's strategies: if an administrative mechanism is used then $\bar{k} = \bar{q}$ is a vector of resource constraints, while in an economic mechanism $\bar{k} = \bar{s}$ is a vector of shares of distribution of the public income, $\sum_{i \in M} s_i = 1$; K is a set of the Principal's strategies which is $K = \prod_{i \in M} [0; r_i]$ in the case of an administrative mechanism and $K = \prod_{i \in M} [0; 1]$ in the case of an economic one; V_i are sets of the agents' strategies. If a mechanism of corruption is used then $V_i = U_i \times [0; 1]$, otherwise $V_i = U_i$; $\bar{J} = (g_0, g_1, \ldots, g_n)$ is a vector of the players' payoffs where g_0 is a social welfare function maximized by the Principal and having the form $g_0 = \sum_{i \in M} p_i(r_i - u_i) + c(\bar{u})$, and $g_{i|i \in M}$ are the agents' payoff functions in the form $g_i = p_i(r_i - u_i) + s_i c(\bar{u})$; Π is a set of control mechanisms used by the Principal. Namely,

$$\Pi = [\pi_a \vee \pi_e] \& [\pi_G \vee \pi_{St}] \& (1 \vee \pi_b).$$

or the set contains administrative and economic mechanisms without a feedback or with it, possibly including corruption. For example, $\Gamma_{hi} =< \{0, 1, 2\}, \bar{k}, \{V_i\}_{i=1}^n, \bar{J}, \{\pi_a \& \pi_G \& \pi_b\} >$ denotes a hierarchical game among Principal and two agents where the Principal uses an administrative mechanism with a feedback and corruption, the Principal's strategy includes resource constraints $\bar{k_i} = q_i(\bar{u}, \bar{b})$, the set of the Principal's feasible strategies is $K = \prod_{i \in M} [0; r_i]$, the set of the agent's feasible strategies is $V_i = U_i \times [0; 1]$ (including a share of bribe), the vector of payoff functions is equal to $\bar{J} = (g_0, g_1, \ldots, g_n)$ and contains the Principal's payoff function $g_0 = \sum_{i \in M} p_i(r_i - u_i) + c(\bar{u})$, and the agents' payoff functions $g_i = p_i(r_i - u_i) + s_i c(\bar{u})$.

Let's introduce in the game (9) an analogue of the price of anarchy (Papadimitriou, 2001). Denote by $NE = \{u_{(1)}^{NE}, ..., u_{(k)}^{NE}\}$ a set of the Nash equilibria in the game (2), $u_{(j)} = (u_{(j)1}, \ldots, u_{(j)n})$ a game outcome, $g_{0\min}^{NE} = \min\{g_0(u_{(1)}^{NE}), \ldots, g_0(u_{(k)}^{NE})\}$, $g_{0\max} = \max_{u \in U} g_0(u) = g_0(u^{\max})$. Then the price of anarchy in the model (9) is

$$PA = \frac{g_{0\min}^{NE}}{g_{0\max}}. \tag{10}$$

It is evident that $PA \leq 1$. If PA is close to one then the efficiency of equilibria is high and the need of coordination in the model (9) is low or absent at all (when $PA = 1$); the lower is PA, the greater is the coordination need.

Two approaches: an empirical one and a theoretical one - can be used in the investigation of an economic control mechanism with a feedback. In the empirical approach the methods of distribution of the public income widely used in practice are analyzed:

$$[\pi_e \& \pi_G]_{emp} = \{s_i = \bar{s}_i(u), s_i \text{ is given}\}.$$

The Principal only fixes a form of the function s, and retires.
An example of the empirical approach is given by the method of proportional allocation when a share of the agent in the public income is proportional to his share in the production of the public good:

$$s_i(u) = \begin{cases} \frac{u_i}{\sum_{j \in M} u_j}, & \exists m : u_m = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

The theoretical approach is based on building the economic mechanism that is optimal for the Principal and considers the interests of agents using Germeier's theorem (Gorelik and Kononenko,1982):

$$[\pi_e \& \pi_G]_{G2} = \{s_i = s_i(u), s_i \text{ is found}\}.$$

An administrative mechanism without a feedback may be implemented in several variants:
1) Principal controls the resource allocation only above, namely, she fixes the amounts q_i such that an agent cannot assign less resources to the public objectives:

$$\pi_a \& \pi_{St} = \{k_i = q_i \in R, i = 1, \ldots, n | 0 \le q_i \le r_i\}.$$

In this case , and a social welfare function has the form

$$g_0 = \sum_{i \in M} p_i(r_i - u_i) + c(\bar{u}) - C(q_1, q_2, \ldots, q_n).$$

2) Principal controls the resource allocation from both sides, and in turn two cases are possible:
(a) the control amounts for the agents are different, namely, the Principal fixes for each agent the thresholds $\overline{q_i}$ and $\underline{q_i}$, such as the agent cannot assign greater or less resources to the public objectives:

$$\pi_a \& \pi_{St} = \{k_i = (\overline{q_i}, \underline{q_i}) \in R^2, i = 1, \ldots, n | 0 \le \underline{q_i} \le \overline{q_i} \le r_i\}.$$

In this case $\underline{q_i} \le u_i \le \overline{q_i}$, and a social welfare function has the form

$$g_0 = \sum_{i \in M} p_i(r_i - u_i) + c(\bar{u}) - C(\overline{q_1}, \underline{q_1}, \overline{q_2}, \underline{q_2}, \ldots, \overline{q_n}, \underline{q_n}).$$

(b) the control amounts for the agents are the same, namely, the Principal fixes the thresholds $\overline{q}$ and $\underline{q}$, such as each agent cannot assign greater or less resources to the public objectives:

$$\pi_a \& \pi_{St} = \{k = (\overline{q}, \underline{q}) \in R^2, i = 1, \ldots, n | 0 \le \underline{q} \le \overline{q} \le 1\}.$$

In this case $\underline{q} r_i \le u_i \le \overline{q} r_i$, and a social welfare function has the form

$$g_0 = \sum_{i \in M} p_i(r_i - u_i) + c(\bar{u}) - C(\overline{q}, \underline{q}).$$

3. System compatibility in the base model

Let's consider the game theoretic model (2) in normal form. It is supposed that:
- function c monotonically increases by all u_i, $c(0,\dots,0)=0$;
- functions p_i monotonically increase by (r_i-u_i) and monotonically decrease by u_i, $p_i(0)=0$ (when $u_i=r_i$);
- if $u_i>0$ then $s_i>0, i=1,\dots,n$. The variant $\sum_{i=1}^n s_i=0$ corresponds to the case when $\forall i\ u_i=0$; then the public income is not produced and there is nothing to share.

Definition 1. A model is system compatible if $PA=1$.

Theorem 1. *Suppose that the functions c and p_i are increasing and concave, $p_i(0)=0$, $c(0)=0$. Then the system compatibility holds if and only if the set of agents consists of two classes: individualists I $(u_i=0)$ and collectivists C $(u_i=r_i)$.*

Proof. Denote

$$x_i=\left(s_ic'\left(\sum_{i=1}^n u_i\right)-p_i'(r_i-u_i)\right)^{-1}(0),$$

$$y_i=\left(c'\left(\sum_{i=1}^n u_i\right)-p_i'(r_i-u_i)\right)^{-1}(0).$$

Then

$$u_i^{NE}=\begin{cases}0,\ x_i<0,\\ x_i,\,0<x_i<r_i,\\ r_i,\ x_i>r_i.\end{cases} \tag{12}$$

$$u_i^{\max}=\begin{cases}0,\ y_i<0,\\ y_i,\,0<y_i<r_i,\\ r_i,\ y_i>r_i.\end{cases} \tag{13}$$

It is seen that the values of strategies coincide on the bounds of the segment $[0,r_i]$, i.e. when the agent is an individualist or a collectivist. Let's prove that the internal values do not coincide. As far $s_i<1$ then $s_ic'\left(\sum_{i=1}^n u_i\right)<c'\left(\sum_{i=1}^n u_i\right)$, therefore, $s_ic'\left(\sum_{i=1}^n u_i\right)-p_i'(r_i-u_i)<c'\left(\sum_{i=1}^n u_i\right)-p_i'(r_i-u_i)$. Denote $f(u_i)=s_ic'\left(\sum_{i=1}^n u_i\right)-p_i'(r_i-u_i)$, $g(u_i)=c'\left(\sum_{i=1}^n u_i\right)-p_i'(r_i-u_i)$. Due to decreasing of the functions $f(u_i)$ and $g(u_i)$ their inverse functions decrease also, and the value of image of the point 0 in the greater function $g(u_i)$ is greater than the image of the point 0 in the smaller function $f(u_i)$. The theorem is proved.

The conditions of Nash equilibrium in the model (2) can be characterized as
$\frac{1}{n}c(r_1,\dots,r_k,0,\dots,0)\geq p_i(r_i), i=1,\dots,k$ (the transition $C\to I$ is not profitable);
$p_j(r_j)\geq\frac{1}{n}c(r_1,\dots,r_k,0,\dots,r_j,\dots,0), i=j+1,\dots,n$ (the transition $I\to C$ is not profitable).

We have

$$g_0^I = g_0(0,\dots,0) = \sum_{j=1}^{n} p_j(r_j),$$
$$g_0^C = g_0(r_1,\dots,r_n) = c(r_1,\dots,r_n),$$
$$g_0^{NE} = g_0(u^{NE}) = c(r_1,\dots,r_k,0,\dots,0) + \sum_{j=k+1}^{n} p_j(r_j),$$

4. Control mechanisms in static CPPI-models

Thus, the condition of system compatibility $PA = 1$ is rarely satisfied by itself, and therefore special control mechanisms are required to provide it. Suppose that maximization of the social welfare (3) is an objective of a specific agent (Principal, leader, social planner, mechanism designer) which has a possibility of impact on the sets of feasible strategies (administrative mechanism) and/or payoff functions (economic mechanism) of other agents to implement the objective. Denote the first possibility by $U_i = U_i(q_i)$, and the second one by $g_i = g_i(p_i, u_i)$. Both types of impact cannot use or use a feedback on control. In the first case a hierarchical game of the type G1 (Stackelberg game), in the second one a hierarchical game of the type G2 (Germeier game) arises (Gorelik and Kononenko,1982). Thus, four types of control mechanisms are possible.

Definition 2. A control mechanism $\mathbf{k}$ in the model (9) is system compatible if the optimal answer of the players $\mathbf{u}(\mathbf{k})$ makes the model system compatible.

Economic control mechanisms π_e in the model (9) are implemented by choosing by the Principal some values s_i:

$$\pi_e = \{k_i = s_i | 0 \le s_i \le 1, \sum_{i \in M} s_i = 1\}.$$

Administrative control mechanisms mean that the Principal can constraint feasible strategies of the agents:

$$\pi_a = \{k_i = (\overline{q_i}, \underline{q_i}) | 0 \le \overline{q_i}, \underline{q_i} \le r_i, \underline{q_i} \le u_i \le \overline{q_i}\}.$$

4.1. Economic mechanisms without a feedback

Suppose that a control mechanism $\pi_e \& \pi_{St} = \{k_i = s_i | 0 \le s_i \le 1, \sum_{i \in M} s_i = 1\}$ is implemented. Using the first order conditions shows that the system compatibility in the interior of the domain of feasible strategies is possible only in the degenerated case. Thus, when $\pi_e \& \pi_{St}$, the system compatibility in the model (9) means as a rule that all agents are individualists or collectivists.

If the condition of system compatibility is not satisfied then it is possible to set a problem of coordination of interests in a weaker form of building an economic control mechanism which maximizes the price of anarchy (10).

4.2. Economic mechanisms with a feedback

Suppose now that a control mechanism $\pi_e \& \pi_{St} = \{k_i = s_i(u) | 0 \le s_i(u) \le 1, \sum_{i \in M} s_i = 1\}$ is implemented in the model. Using the first order conditions shows that the system compatibility in the interior of the domain of feasible strategies is possible only if

$$\frac{\partial s_i(u)}{\partial u_i} = [1 - s_i(u)]\frac{\partial c(u)}{\partial u_i}, i \in M.$$

In the frame of empirical approach $[\pi_e \& \pi_G]_{emp} = \{s_i = \bar{s_i}(u), s_i \text{ is given}\}$ widely spread practical methods of distribution of the public income are investigated. For example, consider a mechanism of proportional allocation (11).

Theorem 2. *A mechanism of proportional allocation is system compatible if and only if the function $c(x)$ is linear.*

Proof. In this case the condition of system compatibility takes the form

$$\sum_{j \neq i} u_j \left[\frac{\partial c\left(\sum_{j\in M} u_j\right)}{\partial u_i} \sum_{j\in M} u_j - c\left(\sum_{j\in M} u_j\right)\right] = 0, i \in M.$$

Let's solve the equation $\frac{\partial c\left(\sum_{j\in M} u_j\right)}{\partial u_i}\sum_{j\in M} u_j - c\left(\sum_{j\in M} u_j\right) = 0$. Transform $\frac{\partial c\left(\sum_{j\in M} u_j\right)}{\partial u_i}\sum_{j\in M} u_j = c\left(\sum_{j\in M} u_j\right)$, $\frac{\partial c\left(\sum_{j\in M} u_j\right)}{c\left(\sum_{j\in M} u_j\right)} = \frac{\partial u_i}{\sum_{j\in M} u_j}$, $\ln c\left(\sum_{j\in M} u_j\right) = \ln \sum_{j\in M} u_j + \hat{c}(u_{-i})$, where $\hat{c}(u_{-i})$ is an integration constant on u_i, $c\left(\sum_{j\in M} u_j\right) = \hat{c}(u_{-i})\sum_{j\in M} u_j$.
The function c depends only on the sum $\sum_{j\in M} u_j$, and $\hat{c}(u_{-i})$ does not depend on the sum. Therefore, $\hat{c}(u_{-i}) = const = c$ that means $c\left(\sum_{j\in M} u_j\right) = c\sum_{j\in M} u_j$. The theorem is proved.

Remind that a function $f(x_1, x_2, ..., x_n)$ is symmetrical relative to the variables x_1, x_2, ..., x_n, if a permutation of any pair of the variables does not change the form of the function, i.e. for any i, j, $1 \le i, j \le n$ holds $f(x_1, x_2, ..., x_i, ..., x_j, ..., x_n) = = f(x_1, x_2, ..., x_j, ..., x_i, ..., x_n)$.

Theorem 3. *An allocation mechanism $s_i(u)$ is system compatible if and only if the function $s_i(u)$ is symmetrical by u_i.*

Proof. The first order conditions are:

$$\begin{cases} \frac{\partial p(u_i)}{\partial u_i} = \frac{\partial c\left(\sum_{j\in M} u_j\right)}{\partial u_i}, \\ \frac{\partial p(u_i)}{\partial u_i} = \frac{\partial s_i(u_i)}{\partial u_i} c\left(\sum_{j\in M} u_j\right) + s_i(u)\frac{\partial c\left(\sum_{j\in M} u_j\right)}{\partial u_i}. \end{cases}$$

or $0 = \frac{\partial s_i(u_i)}{\partial u_i} c\left(\sum_{j\in M} u_j\right) + (s_i(u) - 1)\frac{\partial c\left(\sum_{j\in M} u_j\right)}{\partial u_i}$. Let's transform: $\frac{\frac{\partial c\left(\sum_{j\in M} u_j\right)}{\partial u_i}}{c\left(\sum_{j\in M} u_j\right)} = = \frac{\frac{\partial s_i(u_i)}{\partial u_i}}{1 - s_i(u)}$, $\frac{\partial \ln c\left(\sum_{j\in M} u_j\right)}{\partial u_i} = -\frac{\partial \ln(1 - s_i(u))}{\partial u_i}$, $c\left(\sum_{j\in M} u_j\right) = \frac{\hat{c}(u_{-i})}{1 - s_i(u)}$.
The left hand side is symmetrical by u_i , therefore, the right hand side should also be symmetrical by u_i. It means that $s_i(u)$ is symmetrical by u_{-i} . The theorem is proved.

Notice that the right hand side, therefore $s_i(u)$, depends on $\sum_{j\in M} u_j$. Besides, $s_i(u) = 1 - \frac{\hat{c}(u_{-i})}{c\left(\sum_{j\in M} u_j\right)}$ and consideration of the condition $\sum_{i\in M} s_i = 1$ gives $\sum_{i\in M} s_i(u) = n - \frac{\sum_{i\in M}\hat{c}(u_{-i})}{c\left(\sum_{j\in M} u_j\right)}$, $c\left(\sum_{j\in M} u_j\right) = \frac{\sum_{i\in M}\hat{c}(u_{-i})}{n-1}$.

As far the left hand side $c\left(\sum_{j\in M} u_j\right)$ does not depend on n, the right hand side also does not depend on n. Therefore the denominator $(n-1)$ should be reduced. It is possible only if the sum of n summands in the numerator in the right hand side may be regrouped in $(n-1)$ equal summands that provides the reduction of $(n-1)$. Besides, each of the regrouped summands should depend only on $\sum_{j\in M} u_j$. Thus, the numerator in the right hand side should be presented as $\sum_{i\in M}\hat{c}(u_{-i}) = (n-1)\cdot c\left(\sum_{j\in M} u_j\right)$. Therefore $\hat{c}(u_{-i})$ must be symmetrical by u_{-i}, and finally $\sum_{i\in M}\hat{c}(u_{-i})$ should depend on $\sum_{j\in M} u_j$, i.e. the mechanism $s_i(u)$ may be represented as

$$s_i(u) = 1 - \frac{(n-1)\hat{c}(u_{-i})}{\sum_{i\in M}\hat{c}(u_{-i})}$$

Other economic mechanisms are also possible, for example,

$$s_i(u) = \begin{cases} \frac{1}{|\{j:u_j=r_j\}|}, & u_i = r_i, \\ 0, & \text{otherwise}. \end{cases} \tag{14}$$

This mechanism allocates the public income only among collectivists. Notice that in this case all agents have only two rational strategies: $\forall i : U_i = \{0, r_i\}$, therefore the mechanism (14) reduces a general CPPI-model to the CPPI-model with binary sets of strategies (Gorbaneva and Ougolnitsky, 2015).

Let's now formulate the problem of control mechanisms design in a general form. Suppose a social planner which maximizes the social welfare function (3) reports to all agents the control mechanism

$$s_i(u) = \begin{cases} \frac{1}{|\{j:u_j=u_j^{\max}\}|}, & u_i = u_i^{\max}, \\ 0, & \text{otherwise}, \end{cases}, i = 1, ..., n. \tag{15}$$

Then the agents' payoffs are equal to

$$g_i(u) = \begin{cases} p_i(r_i - u_i^{\max}) + \frac{c(u_i^{\max}, u_{-i})}{|\{j:u_j=u_j^{\max}\}|}, & u_i = u_i^{\max}, \\ p_i(r_i - u_i), & \text{otherwise}. \end{cases}$$

It is evident that in this case $U_i = \{0, u_i^{\max}\}$, because if $u_i > 0$, $u_i \neq u_i^{\max}$ then $g_i(u) = p_i(r_i - u_i) < p_i(r_i)$. Therefore, the mechanism (15) also reduces a general CPPI-model to the model with binary sets of strategies. The Theorem 2 leads to

Corollary 1. *The allocation mechanism*

$$s_i(u) = \begin{cases} \frac{1}{|\{j:u_j=u_j^{\max}\}|}, & u_i = u_i^{\max}, \\ 0, & \textit{otherwise}, \end{cases}, i = 1, ..., n.$$

is not system compatible.

The difficulty is that an i-th player in the moment of decision does not know u_{-i} and respectively $\{j : u_j = u_j^{\max}\}$. Therefore it is difficult to estimate the efficiency of the mechanism (15) (to compare the payoffs) in a general case. It is possible to argue that an optimal answer of the i-th player to the mechanism (15) is

$$u_i^{opt}(s_i) = \begin{cases} u_i^{\max}, \forall u_{-i} \in U_{-i}\ p_i(r_i) \le p_i(r_i - u_i) + \frac{c(u_i^{\max}, u_{-i})}{|\{j : u_j = u_j^{\max}\}|}, \\ 0, \forall u_{-i} \in U_{-i} \qquad p_i(r_i) \ge p_i(r_i - u_i) + \frac{c(u_i^{\max}, u_{-i})}{|\{j : u_j = u_j^{\max}\}|}, \end{cases} \tag{16}$$

i.e. one of the two feasible strategies dominates the other one and is the dominant strategy respectively. But the optimal answer is uncertain if for different u_{-i} the signs of the inequalities are different (i.e. both strategies are non-dominated) (Gorbaneva and Ougolnitsky, 2015). A theoretical approach $[\pi_e \& \pi_G]_{G2} = \{s_i = s_i(u), s_i \text{ is found}\}$ leads to the following result.

Theorem 4. *If functions $p_i(x)$ and $c(x)$ are of power type with a positive exponent less or equal to one then an economic mechanism with a feedback $[\pi_e \& \pi_G]_{G2} = \{s_i = s_i(u), s_i \text{ is found}\}$ is system compatible.*

Due to some special properties of linear functions it is convenient to consider four cases separately:
1) functions $p_i(x)$ and $c(x)$ are linear;
2) functions $p_i(x)$ are linear, and the function $c(x)$ is of power type with a positive exponent less than one;
3) functions $p_i(x)$ are of power type with a positive exponent less than one, and the function $c(x)$ is linear;
4) functions $p_i(x)$ and $c(x)$ are of power type with a positive exponent less than one.
A **proof** is presented for the case of linear functions $p_i(x) = p_i \cdot x$ and $c(x) = c \cdot x$, and is based on the Germeier theorem (Gorelik and Kononenko,1982). Other cases are analyzed similarly.

Proof. Notice that the punishment strategy is $s_i^P = 0$, an i-th agent's optimal answer is $E_i = \{u_i = 0\}$, and his payoff is equal to $L_i = p_i r_i$. The Principal's payoff is equal to

$$K_2 = \max_{s_i} \min_{u_i \in E_i} \left[\sum_{i \in M} p_i(r_i - u_i) + c\left(\sum_{i \in M} u_i \right) \right] = \max_{s_i} \left[\sum_{i \in M} p_i(r_i) \right] = \sum_{i \in M} p_i(r_i)$$

Let's determine a set D_i of such strategies that the i-th player's payoff is greater than L_i:

$$p_i(r_i - u_i) + s_i c\left(\sum_{i \in M} u_i \right) > p_i r_i$$

It is possible only if $s_i > \frac{p_i u_i}{c\left(\sum_{i \in M} u_i\right)}$. The value K_1 is equal to

$$K_1 =$$

$$= \max_{s_i \in D_i} \max_{u_i} \left[\sum_{i \in M} p_i(r_i - u_i) + c\left(\sum_{i \in M} u_i \right) \right] = \max_{u_i} \left[\sum_{i \in M} p_i(r_i - u_i) + c\left(\sum_{i \in M} u_i \right) \right]$$

As it is shown above,

$$u_j^{\max} = \begin{cases} r_j, c > p_j, \\ 0, c < p_j. \end{cases}$$

Let's prove that the inequality $s_i > \frac{p_i u_i}{c(\sum_{i \in M} u_i)}$ can be satisfied that provides a profitability of the strategy $u_j^{\max}$ for the agent. For those agents which have $u_j^{\max} = 0$, it may be provided by the strategy $s_i = \epsilon_i$, for the other agents $s_i > \frac{p_i u_i}{c(\sum_{i \in C} u_i)}$. It holds if $\sum_{i=1}^n s_i(u) = \begin{cases} 1, \exists i : s_i > 0, \\ 0, \forall i : s_i = 0, \end{cases}$ or $\sum_{i \in C} p_i r_i < c\left(\sum_{i \in C} u_i\right)$. As for each summand the inequality $c > p_j$ holds then the condition $\sum_{i \in C} p_i r_i < c\left(\sum_{i \in C} u_i\right)$ is also satisfied, and due to the equivalence of all transforms the initial inequality holds, too. The theorem is proved.

4.3. Administrative mechanisms of system compatibility

Let's suppose that Principal can constraint the sets of feasible strategies of agents. Consider the model (9) with a control mechanism $\pi_a \& \pi_{St} = \{k_i = (\overline{q_i}, \underline{q_i}) \in R^2, i = 1, ..., n | 0 \leq \underline{q_i} \leq \overline{q_i} \leq r_i\}$..
It is clear that if the Principal's possibilities are not bound then the problem of system compatibility has a trivial solution $\underline{q_i} = \overline{q_i} = u_i^{\max}, i \in M$. Therefore, a real setup of the problem requires a consideration of the administrative costs of the Principal. Then her payoff function takes the form

$$g_0(\underline{q}, \overline{q}, u) = \sum_{j \in M} p_j(r_j - u_j) + c(u) - C(\underline{q}, \overline{q}) \to \max, 0 \leq \underline{q_i} \leq \overline{q_i} \leq r_i, i \in M \quad (17)$$

where $C(\underline{q}, \overline{q})$ is a continuously differentiable and convex by all arguments cost function of the Principal. The function increases by $\underline{q}$ and decreases by $\overline{q}$. The function (3) can be considered as a specific case of (18) when $\underline{q} = 0$, $\overline{q} = r = (r_1, ..., r_n)$.

Definition 3. An administrative mechanism $\underline{q}^{\max}(\overline{q}^{\max})$ is weakly compatible if $u = \underline{q}^{\max} \in NE(\underline{q}^{\max})$, and $g_0(\underline{q}, \overline{q}, \underline{q}^{\max}) = \max_{\underline{q}^{\max} \leq u \leq r} g_0(\underline{q}, r, u)$, (respectively, $u = \overline{q}^{\max} \in NE(\overline{q}^{\max})$ and $g_0(\underline{q}, \overline{q}, \overline{q}^{\max}) = \max_{0 \leq u \leq \overline{q}^{\max}} g_0(0, \overline{q}, u)$).

Notice that when a mechanism is weakly compatible then the value of social welfare function is certainly not greater that when the mechanism is system compatible because in the latter case the Principal has no administrative costs.

Theorem 5. *For a weakly compatible administrative mechanism in the model (9) with the Principal's payoff function (17)* $\underline{q_i}^{\max} < u_i^{\max} < \overline{q_i}^{\max}$, $i \in M$ *holds.*

Proof. From (17) follows that to find $u_i^{\max}$ it is required to solve the system of equations $-p_i'(r_i - u_i) + c'(u) = 0$, to find $\underline{q_i}^{\max}$ it is required to solve the system of equations $-p_i'(r_i - \underline{q_i}) + c'(\underline{q}) = C'_{\underline{q_i}}(\underline{q}, \overline{q}) > 0$, and to find $\overline{q_i}^{\max}$ it is required to solve the system of equations $-p_i'(r_i - \overline{q_i}) + c'(\overline{q}) = C'_{\overline{q_i}}(\underline{q}, \overline{q}) < 0$. In all three cases the same function in the left hand side decreases, and the values in the right hand side are strictly ordered. Therefore, $\underline{q_i}^{\max} < u_i^{\max} < \overline{q_i}^{\max}$. The theorem is proved.

Corollary 2. *It is senseless for the Principal to constraint the agents from above.*

The **proof** follows from the Theorem 1, in which it is shown that $u_i^{NE} < u_i^{\max}$, therefore, due to Theorem 5, $u_i^{NE} < \overline{q_i}^{\max}$. Thus, in the condition $\underline{q_i} \le u_i \le \overline{q_i}$ the right inequality is satisfied automatically.
So, let's consider the mechanism $\pi_a \& \pi_{St} = \{k_i = q_i \in R, i = 1, ..., n | 0 \le q_i \le r_i, q_i \le u_i \le r_i\}$. with Principal's payoff function $g_0(q, u) = \sum_{j \in M} p_j(r_j - u_j) + c(u) - C(q) \to \max$, in which the Principal sets only the left bound of the constraints.

Theorem 6. *An administrative mechanism $\pi_a \& \pi_{St} = \{k_i = q_i \in R | 0 \le q_i \le r_i, q_i \le u_i \le r_i\}$. in the model (9) is system compatible.*

Proof. The optimal strategy of an agent in the model (9) without consideration of the condition $q_i \le u_i \le r_i$ is u_i^{NE} , calculated by the formula (12), and the strategy optimal for the Principal is $q_i = u_i^{\max}$, calculated by the formula (13). In Theorem 1 it is proved that $u_i^{\max} > u_i^{NE}$, therefore the agent's optimal strategy with consideration of the condition $q_i \le u_i \le r_i$ is $u_i = q_i = u_i^{\max}$ that means system compatibility. The theorem is proved.

The following interpretation of the result is possible: if a one-side constraint on the resource allocation from below is costless then the Principal can compel the agents to make the desirable decision.

Theorem 7. *If Principal's payoff function has the form (17) then a control mechanism $\pi_a \& \pi_{St} = \{k_i = q_i \in R | 0 \le q_i \le r_i, q_i \le u_i \le r_i\}$. in the model (9) is weakly compatible if for any one of the two conditions is satisfied:*

$$Arg \max_{u_i \in R} \left[p_i(r_i - u_i) + s_i c \left(\sum_{i \in M} u_i \right) \right] \le$$

$$\le Arg \max_{q_i \in R} \left[\sum_{j \in M} p_j(r_j - q_j) + c \left(\sum_{i \in M} q_i \right) - C(q) \right].$$

or

$$Arg \max_{u_i \in R} \left[p_i(r_i - u_i) + s_i c \left(\sum_{i \in M} u_i \right) \right] > r_i,$$

$$Arg \max_{q_i \in R} \left[\sum_{j \in M} p_j(r_j - q_j) + c \left(\sum_{i \in M} q_i \right) - C(q) \right] > r_i.$$

Notice that a weak compatibility is also possible only on the bounds of the segment q_i, r_i (in the case of an administrative mechanism). Therefore to achieve the weak compatibility it is necessary to have a partition of the set of agents on individualists ($u_i = q_i$) and collectivists ($u_i = r_i$).

Proof. Let's find a Nash equilibrium. An optimal agent's answer to Principal's strategy is

$$u_i^{NE} = \begin{cases} u_i^*, \, q_i < u_i^* < r_i, \\ q_i, \; q_i > u_i^*, \\ r_i, \; u_i^* > r_i, \end{cases}$$

where $u_i^* = Arg\max_{u_i\in R}\left[p_i(r_i-u_i)+s_ic\left(\sum_{i\in M}u_i\right)\right]$ (without consideration of the constraints $q_i \le u_i \le r_i$). From the point of view of the Principal the agent's optimal strategy is

$$u_i^{\max} = \begin{cases} u_i^{**},\, q_i < u_i^{**} < r_i, \\ q_i, \quad q_i > u_i^{**}, \\ r_i, \quad u_i^{**} > r_i, \end{cases}$$

where $u_i^{**} = Arg\max_{u_i\in R}\left[\sum_{j\in M}p_j(r_j-u_j)+c\left(\sum_{i\in M}u_i\right)-C(q)\right]$ (similarly). Notice that there are unique values u_i^* and u_i^{**} due to negativity of the second derivatives of the functions g_i and g_0, and it is proved in Theorem 1 that $u_i^{\max} \ge u_i^{NE}$.

If it is profitable for an agent to be a collectivist ($u_i = r_i$), the Principal has no need to provide control, therefore, $q_i = 0$. In this case the function g_0 decreases by q_i.

If an agent is neither individualist nor collectivist ($q_i < u_i < r_i$) then no control is also required, therefore again $q_i = 0$.

If it is profitable for an agent to be an individualist ($u_i = q_i$) then the optimal q_i is found as a solution of the problem $q_i^* = Arg\max_{q_i\in R}\left[\sum_{j\in M}p_j(r_j-q_j)+c(q)-\right.$ $\left.-C(q)\right]$, and with consideration of the constraints

$$q_i = \begin{cases} q_i^*,\, 0 < q_i^* < r_i, \\ 0, \;\; 0 > q_i^*, \\ r_i,\; q_i^* > r_i, \end{cases}$$

The theorem is proved.

In presence of a cost on a one-side control from below in resource allocation a set of agents is divided on three subsets: (1) those who are collectivists independently on the Principal's interests (the set I_1); (2) those who assign for the production of public good less resources than the Principal wants but she can't prevent it (the set I_2); (3) those who would like to assign for the production of public good less resources than the Principal wants and she can increase this value (the set I_3). The Principal can impact only on the elements of the third subset but she cannot always provide the system compatibility.

Also, a case may be considered when $\forall i\; q_i = q$. As amounts r_i are different, in this case the measurement should be done in shares (not amounts) of resources. The Principal's optimization problem has the form

$$g_0(q,u) = \sum_{j\in M}p_j(r_j-u_j)+c(u)-C(q)\to\max, 0\le q\le 1, i\in M$$

To find the optimal values q it is required to solve the equation

$$-\sum_{i\in I_3}r_jp_j'(r_j(1-q))+c'\left(q\sum_{j\in I_1}r_j+\sum_{j\in I_2}u_j^*\right)\sum_{j\in I_1}r_j - C'(q) = 0. \qquad (18)$$

5. Conclusuion

The paper is dedicated to the investigation of the static game theoretic models of coordination of private and public interests (CPPI-models) in resource allocation, to the revealing of conditions of system compatibility in these models, and to the analysis of control mechanisms that permit to attain or approach the system compatibility. The system compatibility means that in any Nash equilibrium a function of social welfare attains its global maximum. Two control mechanisms are considered: an administrative one (compulsion) and an economic one (impulsion), each of them with or without a feedback on control. The following conclusions can be made. The system compatibility in a base CPPI-model without control is possible if and only if all players are individualists (they assign all resources to their private activity) or collectivists (they assign all resources for the production of a public good). If a function of allocation of the public income among players is given then the system compatibility is possible if and only if the function is symmetrical relative to for each player i. If this function is to be found then the system compatibility may be provided if and only if the functions of private and public income are of power type with a positive exponent less or equal than one. As for administrative control mechanisms, the system and weak compatibility are to be differentiated. The weak compatibility means that the system compatibility is attained when the values of constraints are chosen as strategies. If control costs are absent then the system compatibility is reachable, otherwise only the weak compatibility may be provided in the case when all agents are individualists or collectivists. It is also shown that in the model (17) it is sufficient to the Principal to constraint only the agents' individualism, i.e. to use the bounds from below.

References

Algorithmic Game Theory (2007). Ed. by N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani. Cambridge University Press.

Bergstrom, T., C. Blume and H. Varian (1986). *On the private provision of public goods* . Journal of Public Economics, **29**, 25–49.

Boadway, R., P. Pestiau and D. Wildasin (1989a). *Non-cooperative behavior and efficient provision of public goods.* Public Finance, **44**, 1–7.

Boadway, R., P. Pestiau and D. Wildasin (1989b). *Tax-transfer policies and the voluntary provision of public goods.* Journal of Public Economics, **39**, 157-176.

Burkov, V. N., V. I. Opoitsev (1974). *A Meta-game Approach to the Control in Hierarchical Systems.* Automation and Remote Control, **35(1)**, 92–103.

Christodoulou, G., A. Sgouritza, B. Tang (2015). *On the Efficiency of the Proportional Allocation Mechanism for Divisible Resources.* M. Hoefer (Ed.):SAGT, LNCS 9347, 165–177.

Germeier, Yu. B., I. A. Vatel (1975). *Equilibrium situations in games with a hierarchical structure of the vector of criteria* . Lecture Notes on Computer Science, **27**, 460–465.

Gorbaneva, O. I., G. A. Ougolnitsky (2013). *Purpose and Non-Purpose Resource Use Models in Two-Level Control Systems.* Advances in Systems Science and Applications, **13(4)**, 379–391.

Gorbaneva, O. I., G. A. Ougolnitsky (2015). *System Compatibility, Price of Anarchy and Control Mechanisms in the Models of Concordance of Private and Public Interests.* Advances in Systems Science and Applications, **15(1)**, 45–59.

Gorbaneva, O. I., G. A. Ougolnitsky, A. B. Usov (2016). *Modeling of Corruption in Hierarchical Organizations.* N.Y.: Nova Science Publishers.

Gorelik, V. A., A. F. Kononenko (1982). *Models of Decision Making in Ecological-Economic Systems.* M. (in Russian)

Kahana, N., D. Klunover (2016). *Private provision of a public good with a time-allocation choice.* Social Choice and Welfare, **47**, 379–386.

Kukushkin, N. S. (1994). *A Condition for Existence of Nash Equilibrium in Games with Private and Public Objectives*. Games and Economic Behavior, **7**, 177–192.

Laffont, J.J., D. Martimort (2002). *The Theory of Incentives. The Principal-Agent Model*. Princeton University Press.

Mechanism design and management: mathematical methods for smart organizations (2013). Ed. by D. Novikov. New York: Nova Science Publishers.

Novikov, D. (2013). *Theory of Control in Organizations*. New York: Nova Science Publishers.

Papadimitriou, C. H. (2001). *Algorithms, games, and the Internet*. Proc.33th Symposium Theory of Computing, 749–753.

Warr, P. (1983). *The private provision of a public good is independent of the distribution of income*. Economics Letters, **13**, 207–211.