Contributions to Game Theory and Management, X, 27–41

Information Pooling Game in Multi-Portfolio Optimization*

Jing Fu

Fukuoka Institute of Technology,
Department of System Management,
3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka, 811-0295 Japan
E-mail: j.fu@fit.ac.jp

Abstract In this paper, an information pooling game is proposed and studied for multi-portfolio optimization. Our approach differs from the classical multi-portfolio optimization in several aspects, with a key distinction of allowing the clients to decide whether and to what extent their private trading information is shared with others, which directly affects the market impact cost split ratio. We introduce a built-in factor related to the clients' vertical fairness regarding the outcomes, which is termed as "dissatisfaction indicator". With balanced horizontal dissatisfactions across all accounts, the main formulation guarantees that no client is systematically advantaged or disadvantaged by the information pooling process. This is a novel mechanism to incorporate both the horizontal and vertical fairness in the optimization process. We show that information pooling solution outperforms the pro-rata collusive solution from fairness aspect, and the Cournot-Nash equilibrium solution for its Pareto optimality. Moreover, the empirical results suggest that within our framework, information pooling has non-negative impact on all participants' perceived fairness, although it may hurt some account's realized benefit compared to null information pool.

Keywords: information pooling, multi-portfolio optimization, horizontal fairness, vertical fairness

1. Introduction

Since the introduction of modern portfolio theory (Markowitz, 1952), financial models with incorporation of various new factors and findings have been constantly reinvented. In the portfolio optimization process, almost all portfolios need to be adjusted during their lifetimes, so incurring periodic transaction costs is inevitable. In October 2000, the Texas Permanent School Fund rebalanced its portfolio of 2,200 securities of about \$17.5 billion. Not to mention the administrative costs, the transaction cost itself is \$120 million (PlexusGroup, 2002). Managers cannot afford to ignore transaction costs, a large portion of which is attributed to the market impact.

In practice, financial advisers usually provide their services to multiple clients simultaneously. In order to efficiently serve a large number of clients, Securities and Exchange Commission (SEC) allows the manager to "bunch orders on behalf of two or more client accounts, so long as the bunching is done for the purpose of achieving best execution, and no client is systematically advantaged or disadvantaged" (Securities and Exchange Commission, 2011). In this case, a problematic interaction arises between the multiple portfolios because the transaction cost for a given client may depend on the overall level of trading and not just on that client's trading requirements (O'Cinneide et al., 2006). The rebalancing price tends to be underestimated largely due to the market impact of bunched trades: benefits sought for

* This work was supported by start-up grants of Fukuoka Institute of Technology.

individual accounts through trading are lost due to increased overall transaction costs.

To model the market impact cost more accurately, several multi-portfolio optimization approaches have been developed. In the first paper recognizing this problematic interaction (O'Cinneide et al., 2006), the authors propose a pro-rata collusive solution to the problem where the objective function is the sum of the objective functions of each individual account. They assert that this is fair since the solution obtained is the same as if each account directly competes in an open market for liquidity. Showing that certain accounts may be better off acting alone instead of participating in the collusive solution, practitioners propose to solve the problem by identifying the set of portfolios that form a Cournot-Nash equilibrium (Savelsbergh et al., 2010). In their model, each account is optimizing its own objective by assuming that it is "made aware" of the trades of other accounts that are being pooled together for execution, and gives its best response. An attractive property of this approach is that the actual market impact cost for each account is exactly what it has anticipated. However, the Cournot-Nash solution is not necessarily Pareto optimal, which means that it may violate the SEC best execution rules. Moreover, some heuristic approach has to be applied to bypass the intractability problem in solving the overall equilibrium (Fabozzi et al., 2010). A recent publication (Iancu and Trichakis, 2014) has well documented the solutions available to multi-portfolio optimization problems.

Let us organize the fairness issue from the investors' perspective. The contract renewal decision by a client is not only affected by the final gain/loss of the investment, but also by the performance difference between her own account and others' (horizontal fairness), and the difference between her expected and realized net return (vertical fairness). Investors care about fairness, for it is a crucial role in establishing and maintaining relationships (Kahneman et al., 1986). However, it is frequently sacrificed in the efficient approaches, and the perfectly fair Cournot-Nash equilibrium solution is not Pareto optimal. Hence, a natural question arises: how can we implement both horizontal and vertical fairness in an efficient multi-portfolio optimization solution reasonably?

Vertical fairness is responsive to a wide range of factors, i.e., needs, wants, beliefs, prior expectations, etc. An empirical study on procedural fairness (Bies et al., 1993) suggests that without involvement (voice) of investors, the portfolio optimization process is less than appropriate to be regarded as fair. This indicates that an investor participates more in the optimization process, more information is acquired, and the vertical fairness might be improved. We propose an information pooling game to give a potential answer to the above-mentioned question.

The remainder of this paper is organized as follows. In Section 2, three related prominent solutions in literature are reviewed and discussed. In Section 3, key modeling choices in information pooling game is elaborated in detail first, and the main formulation is highlighted then. In Section 4, to compare the different solution concepts and verify the effect of information pooling, two numerical studies are conducted and discussed. Finally, Section 5 summarizes the main contribution of this work and comments on future research briefly.

2. Multi-portfolio Optimization

In this section, three prominent approaches proposed in literature to solve the multi-portfolio optimization problem are reviewed and discussed. Suppose a financial adviser is managing n distinct portfolios (accounts), indexed by $\boldsymbol{P} = \{1, ..., n\}$. To improve the operational efficiency, managers prefer to invest in assets from the same pool of available investments, reflecting a particular investment style. Thus, for simplicity, the pool of investments available for all clients is assumed to be $\boldsymbol{A} = \{1, ..., m\}$. In this paper, each account is assumed to have an all-cash initial position to simplify the discussion. In other words, in a single rebalancing period, all clients are not allowed to short assets. Let $\boldsymbol{w} = (w_1, ..., w_n) \in \mathbb{R}^n$ denote the initial cash positions of all accounts. Let $\boldsymbol{x_j} \in \mathbb{R}^m$ represent the rebalancing trades of account $j \in \boldsymbol{P}$ in units of currency, and $\boldsymbol{x = (x_1, ..., x_n)} \in \mathbb{R}^{mn}$ represent the vector of all trades. Natural constraints in a single rebalancing period to $\boldsymbol{x_j}$ are $\boldsymbol{x_j} \geq \boldsymbol{0}$ and $\sum_{i \in \boldsymbol{A}} x_{ij} \leq w_j$. Let $\xi_j \subseteq \mathbb{R}^m$ denote the feasible trades of account j satisfying the two constraints above.

Market impact cost is, broadly speaking, the price an investor has to pay for obtaining the liquidity in the market. It is the deviation of the transaction price from the market (mid) price that would have prevailed had the trade not occurred. In general, liquidity providers experience negative costs while liquidity demanders will face positive costs. One of the common approaches in both literature and practice to model market impact is through a nonlinear and strictly convex function of the amounts traded in the form $\boldsymbol{x}^T c(\boldsymbol{x})$, where $c(\boldsymbol{x}) = (c_1(\boldsymbol{x}), ..., c_m(\boldsymbol{x}))$ is a vector function giving the cost per unit traded for each asset. The vector function $c(\boldsymbol{x})$ is assumed to be independent for each asset (ignoring the cross-asset price impact) and expressed in the form of polynomial $c_i(\boldsymbol{x}) = (\sum_{j \in \boldsymbol{P}} x_{ij})^p, \ \forall i \in \boldsymbol{A}$, where p is a rational number between 0.5 and 1 (Almgren et al., 2005). Then, the market impact cost of executing the trades of account $j \in \boldsymbol{P}$ is given as $\boldsymbol{x_j}^T c(\boldsymbol{x})$.

2.1. Independent Optimization Solution

First, consider a simplest setting with a single account $j \in \boldsymbol{P}$, where the objective function for account j is to maximize its net utility. The independent optimization problem to determine the portfolio for account j can be represented as

$$\max_{\boldsymbol{x_j} \in \xi_j} \ u_j(\boldsymbol{x_j}) - \boldsymbol{x_j}^T c(\boldsymbol{x_j}) \tag{1}$$

The net utility for portfolio j is the expected return $u_j(\boldsymbol{x_j})$ derived from its rebalancing trades, subtracting its market impact cost $\boldsymbol{x_j}^T c(\boldsymbol{x_j})$. Notice that $u_j(\boldsymbol{x_j})$ is assumed to be concave, $\boldsymbol{x_j}^T c(\boldsymbol{x_j})$ is convex, ξ_j is a convex subset of $\mathbb{R}^m$. Via convex optimization techniques, the problem can be solved efficiently in variables $\boldsymbol{x_j}$. The independent optimization problem has been extensively studied in the literature.

A direct expression for the problem above is that each account is acting independently and unaware of the market impact by the trades of other accounts. Let us denote the optimal solution as $(\boldsymbol{x_j^{ind}})_{j \in \boldsymbol{P}}$. If all accounts are optimized following this model, the true market impact cost to each account is $(\boldsymbol{x_j^{ind}})^T c(\boldsymbol{x_1^{ind}, ..., x_n^{ind}})$, which is greater than the prior expectation $(\boldsymbol{x_j^{ind}})^T c(\boldsymbol{x_j^{ind}})$ and reduces the vertical fairness of the clients considerably. A numerical example has shown that the true market impact cost to each account might have been 9900% higher than the anticipation in a 100-portfolio setting (Savelsbergh et al., 2010). Neither horizontal nor vertical fairness can be implemented in independent optimization solution.

2.2. Collusive Solution

The basic idea of the collusive solution is to optimize all trades simultaneously by aggregating the utility functions of all accounts (O'Cinneide et al., 2006). The problem can be written as

$$\max_{\boldsymbol{x_j} \in \xi_j, \forall j \in \boldsymbol{P}} \sum_{j \in \boldsymbol{P}} u_j(\boldsymbol{x_j}) - (\sum_{j \in \boldsymbol{P}} x_{ij})^T_{i \in \boldsymbol{A}} c(\boldsymbol{x}) \tag{2}$$

As with the independent optimization solution, collusive optimization problem can be solved efficiently as well. The resulting market impact cost is allocated proportionally to the trading amount of each account, hence we also refer the original collusive solution as pro-rata collusive solution in this paper. The authors argue that this approach is horizontally fair as the solution $(\boldsymbol{x_j^{col}})_{j \in \boldsymbol{P}}$ is the same as the one that would have been obtained if clients are competing in an open market for liquidity. However, in certain situations, some accounts may have to sacrifice their own benefits for the good of others in order to maximize the total welfare (Savelsbergh et al., 2010), and the horizontal fairness cannot be justified if those accounts deviate from the collusive solution.

2.3. Cournot-Nash Equilibrium Solution

Motivated by the significant underestimation of market impact in independent optimization and unfairness in the collusive solution, an equilibrium solution is developed by optimizing each account's objective with the assumption that the trade decisions of all other accounts that participate in the pooled trading have been made and fixed (Savelsbergh et al., 2010). More precisely, for account $j \in \boldsymbol{P}$, the trades for all other accounts $\boldsymbol{x_{-j}} = (\boldsymbol{x_k} : k \neq j \in \boldsymbol{P})$ are fixed and known. The optimization problem for j can be modeled as

$$\max_{\boldsymbol{x_j} \in \xi_j} \; u_j(\boldsymbol{x_j}) - \boldsymbol{x_j}^T c(\boldsymbol{x_j}, \boldsymbol{x_{-j}}) \tag{3}$$

In microeconomics, the equilibrium solution $(\boldsymbol{x_j^{cn}})_{j \in \boldsymbol{P}}$ is referred to as Cournot-Nash equilibrium (Mas-Colell, 1984). It has the property that the expected market impact cost exactly corresponds to the realized market impact cost for each account and no client will have the incentive to deviate from her Cournot-Nash portfolio unilaterally, which is superior to the previous two approaches from the vertical fairness facet. However, Cournot-Nash solution is not necessarily Pareto optimal. It is possible to have at least one account improved without negatively impacting any other account with the independent optimization approach, violating the best execution rules.

3. Information Pooling Game

In this section, a multi-portfolio optimization approach with incorporation of both horizontal and vertical fairness from the clients' perspective is proposed and justified, which is termed as information pooling game. First, three key modeling choices will be explained, namely,

1) information pooling: it allows the clients to decide whether and to what extent their private trading information is shared with others in the same bunched trade;
2) vertical fairness: for each account, it is reflected by the difference between the expected and realized net utility, which is termed as dissatisfaction indicator;
3) horizontal fairness: information pooling process does not inflict particularly high or low dissatisfaction to any account.

3.1. Information Pooling

Cournot-Nash equilibrium is criticized to be unfair from certain aspect, because it coerces the clients to participate in an "artificial game" and share their complete information with others (Iancu and Trichakis, 2014). Motivated by the shortcomings of Cournot-Nash equilibrium, our approach invites all clients to decide whether and to what extent to share their private trading information. Moreover, as is discussed in the introduction, initiative and active participation in the portfolio optimization process improves the information transparency, and ultimately the perceived fairness of clients.

For a more detailed explanation, let $\boldsymbol{\tau} = (\boldsymbol{\tau_j})_{j \in \boldsymbol{P}} \in \mathbb{R}^{mn}$ denote an information pool from all clients, where $\boldsymbol{\tau_j} = (\tau_{ij})_{i \in \boldsymbol{A}} \in \mathbb{R}^m$ is a binary vector and $\tau_{ij} \in \{0, 1\}$ is a binary indicator of client j's willingness or preference on whether to share her trading information of asset i. More precisely, $\tau_{ij} = 0$ indicates that client j rejects to pool her trading information of asset i, while $\tau_{ij} = 1$ indicates that j is willing to share i's trading information with others who contribute to the information pool of i. It is a natural and fair assumption to avoid the free-rider phenomenon in information pooling, and we define the vector function of the cost per unit traded for each asset as below

Definition 1. For account $k \in \boldsymbol{P}$, its estimation on the vector function of the cost per unit traded for asset $i \in \boldsymbol{A}$ with information pool $\boldsymbol{\tau}$ can be defined as $c^k(\boldsymbol{x}|\boldsymbol{\tau}) = (c_i^k(\boldsymbol{x}|\boldsymbol{\tau}))_{i \in \boldsymbol{A}}$, where

$$c_i^k(\boldsymbol{x}|\boldsymbol{\tau}) = \begin{cases} (x_{ik})^p & if\ \tau_{ik} = 0 \\ (\sum_{j \in \boldsymbol{P}}(\tau_{ij} x_{ij}))^p & if\ \tau_{ik} = 1 \end{cases} \tag{4}$$

Let $U_j^\varepsilon(\boldsymbol{x_j^{ip}}|\boldsymbol{\tau})$ denote the expected net utility for account $j \in \boldsymbol{P}$ with information pool $\boldsymbol{\tau}$. It can be derived by the information pooling problem as follows

$$\{U_j^\varepsilon(\boldsymbol{x_j^{ip}}|\boldsymbol{\tau}) = \max_{\boldsymbol{x_j} \in \xi_j} \ u_j(\boldsymbol{x_j}) - \boldsymbol{x_j}^T c^j(\boldsymbol{x}|\boldsymbol{\tau})\} \tag{5}$$

Conditional on a fixed information pool $\boldsymbol{\tau}$, the set of equilibrium solutions to problem (5) is the same as the set of simultaneous solutions of the first-order optimality conditions for all accounts. For account $j \in \boldsymbol{P}$, let $U_j^\varepsilon(\boldsymbol{x_j}|\boldsymbol{\tau}) = u_j(\boldsymbol{x_j}) - \boldsymbol{x_j}^T c^j(\boldsymbol{x}|\boldsymbol{\tau})$, and the conditions can be written as

$$\begin{aligned}
-\frac{\partial U_j^{\varepsilon}(\boldsymbol{x_j}|\boldsymbol{\tau})}{\partial x_{ij}} = -\frac{\partial u_j(\boldsymbol{x_j})}{\partial x_{ij}} + c_i^j(\boldsymbol{x}|\boldsymbol{\tau}) + x_{ij}\frac{\partial c_i^j(\boldsymbol{x}|\boldsymbol{\tau})}{\partial x_{ij}} &\geq 0, \ \forall i \in \boldsymbol{A} \\
x_{ij}(-\frac{\partial U_j^{\varepsilon}(\boldsymbol{x_j}|\boldsymbol{\tau})}{\partial x_{ij}}) &= 0, \ \forall i \in \boldsymbol{A} \\
x_{ij} &\geq 0, \ \forall i \in \boldsymbol{A} \\
\sum_{i\in\boldsymbol{A}} x_{ij} &\leq w_j
\end{aligned} \tag{6}$$

This is a nonlinear complementary problem $NCP(\mathbb{R}_+^m, -\nabla U_j^{\varepsilon}(\boldsymbol{x_j}|\boldsymbol{\tau}))$ such that

$$\boldsymbol{x_j} \geq \boldsymbol{0}, \ -\nabla U_j^{\varepsilon}(\boldsymbol{x_j}|\boldsymbol{\tau}) \geq \boldsymbol{0}, \ (\boldsymbol{x_j})^T(-\nabla U_j^{\varepsilon}(\boldsymbol{x_j}|\boldsymbol{\tau})) = 0 \tag{7}$$

with and an extra constraint of $\sum_{i\in\boldsymbol{A}} x_{ij} \leq w_j$.

Lemma 1. *$(\boldsymbol{x_j^{ip}})_{j\in\boldsymbol{P}}$ is an equilibrium of the information pooling problem if and only if $(\boldsymbol{x_j^{ip}})_{j\in\boldsymbol{P}}$ is a solution to $\{NCP(\mathbb{R}_+^m, -\nabla U_j^{\varepsilon}(\boldsymbol{x_j}|\boldsymbol{\tau}))\}_{j\in\boldsymbol{P}}$ constrained by upper bounded trades of $\sum_{i\in\boldsymbol{A}} x_{ij} \leq w_j, \ \forall j \in \boldsymbol{P}$.*

In portfolio selection theory, the expected return for account $j \in \boldsymbol{P}$ in a single rebalancing period is generally modeled as $u_j(\boldsymbol{x_j}) = \boldsymbol{\varpi}^T\boldsymbol{x_j}$, where $\boldsymbol{\varpi} = (\varpi_i)_{i\in\boldsymbol{A}}$ is a random return rate vector. We prove that

Theorem 1 (Existence and Uniqueness of Equilibrium Solution). *Assume $u_j(\boldsymbol{x_j}) = \boldsymbol{\varpi}^T\boldsymbol{x_j}, \ \forall j \in \boldsymbol{P}$, then there exists a unique equilibrium solution $(\boldsymbol{x_j^{ip}})_{j\in\boldsymbol{P}}$ to the information pooling problem.*

Remark 1. With an null information pool where all accounts deny to share their trading information, $(\boldsymbol{x_j^{ip}})_{j\in\boldsymbol{P}}$ corresponds to the independent optimization solution $(\boldsymbol{x_j^{ind}})_{j\in\boldsymbol{P}}$. In complete information pooling where all accounts reach a consensus on sharing their trading information, $(\boldsymbol{x_j^{ip}})_{j\in\boldsymbol{P}}$ is consistent with the Cournot-Nash equilibrium solution $(\boldsymbol{x_j^{cn}})_{j\in\boldsymbol{P}}$.

3.2. Vertical Fairness

The clients are invited to express their preferences on pooling the trading information, however, this approach does not affect the aggregative optimization by the manager for efficiency and best execution. In other words, the bunched trading decisions are still made in accordance to the collusive solution, and this paper concerns the split mechanism of the resulting market impact cost. Let $r_j^{\boldsymbol{\tau}}$ denote the market impact cost split ratio of account $j \in \boldsymbol{P}$ with information pool $\boldsymbol{\tau}$. Let $\rho_j^{\boldsymbol{\tau}} \subseteq (0,1)$ denote the feasible set of j's split ratio satisfying $r_j^{\boldsymbol{\tau}} \geq 0$ and $\sum_{j\in\boldsymbol{P}} r_j^{\boldsymbol{\tau}} = 1$. Then the realized net utility of account j with information pool $\boldsymbol{\tau}$ can be written as

$$U_j(\boldsymbol{x_j^{col}}|\boldsymbol{\tau}) = u_j(\boldsymbol{x_j^{col}}) - r_j^{\boldsymbol{\tau}}(\sum_{j\in\boldsymbol{P}} x_{ij}^{col})_{i\in\boldsymbol{A}}^T c(\boldsymbol{x^{col}}) \tag{8}$$

where $(\boldsymbol{x_j^{col}})_{j\in\boldsymbol{P}}$ is the solution to the optimization problem (2). Following the argument on perceived fairness such that actions which made some party worse off than the prior expectations are generally viewed as unfair (Kahneman et al., 1986), we have

Definition 2. A dissatisfaction indicator of account $j \in \boldsymbol{P}$ with information pool $\boldsymbol{\tau}$ is defined as

$$ds_j^{\boldsymbol{\tau}} = \frac{U_j^{\varepsilon}(\boldsymbol{x_j^{ip}}|\boldsymbol{\tau}) - U_j(\boldsymbol{x_j^{col}}|\boldsymbol{\tau})}{U_j^{\varepsilon}(\boldsymbol{x_j^{ip}}|\boldsymbol{\tau})} \tag{9}$$

The dissatisfaction indicator reflects the vertical fairness for a client by the relative difference between her expected and realized net utility, and higher dissatisfaction implies worse vertical fairness.

3.3. Horizontal Fairness

From the clients' perspective, they are desiring that any trade $\boldsymbol{x}$ executed with information pool $\boldsymbol{\tau}$ will generate non-positive dissatisfaction. However, it is very difficult to be implemented both theoretically and in practice. Hence, the following optimization problem is introduced to decide the market impact cost split ratio.

$$\min_{r_j^{\boldsymbol{\tau}} \in \rho_j^{\boldsymbol{\tau}}, \forall j \in \boldsymbol{P}} Var(ds_1^{\boldsymbol{\tau}}, ds_2^{\boldsymbol{\tau}}, ..., ds_n^{\boldsymbol{\tau}}) \tag{10}$$

The optimal solution $(r_j^{\boldsymbol{\tau}*})_{j \in \boldsymbol{P}}$ guarantees the horizontal fairness in splitting the resulting market impact cost by minimizing the variance of dissatisfaction indicators across all accounts. Although 100% envy-freeness is not implemented in our mechanism, at least clients' dissatisfactions (or satisfactions) do not spread out too much from a certain level. For example, an investor thinks it unfair if she suffers a 10% dissatisfaction while others in the same bunched trade only suffer 1%, however, it will be judged to be fair if others are dissatisfied at 9.99%.

3.4. Main Formulation: Information Pooling Game

Next the main formulation based on the modeling choices elaborated above will be summarized. The manager would proceed as follows

1) Determine the trades $\boldsymbol{x^{col}}$ by solving the collusive optimization problem (2), and execute it.
2) Invite each client $j \in \boldsymbol{P}$ to determine her information pooling strategy $\boldsymbol{\tau_j} = (\tau_{ij})_{i \in \boldsymbol{A}}$, which forms an information pool $\boldsymbol{\tau}$.
3) Authorize client j to access her corresponding information pool. Both the manager and client j may estimate j's expected net utility by solving the information pooling problem (5) with SLCP (Sequential Linearly Constrained Programming).
4) Determine the split ratio $(r_j^{\boldsymbol{\tau}})_{j \in \boldsymbol{P}}$ of the resulting market impact cost by solving problem (11), where the dissatisfaction indicator for account $j \in \boldsymbol{P}$ is defined by equation (10). Then the realized net utility for j can be determined by equation (9).

From clients' perspective, the process above can be viewed as an information pooling game. They have to decide their own information pooling strategies, and the formed information pool directly affects the allocation of resulting market impact cost by the bunched trades. Here is a very simple example illustrating our mechanism.

Example 1 (Information Pooling Game). Suppose that there are only two accounts, account 1 with \$100 and account 2 with \$10 initially for investment. Furthermore, suppose there is only one risky asset available for investment with an expected return rate, i.e., 40%. Assume $p = 0.6$ ($p = 0.6 \pm 0.038$ with 67% probability, Almgren et al., 2005), then

1) The collusive solution is $\boldsymbol{x^{col}} = (0.0583, 0.0408)$, and the total resulting market impact cost is 0.0248.
2) The potential information pools are $\boldsymbol{\tau^1} = (0, 0)$, $\boldsymbol{\tau^2} = (0, 1)$, $\boldsymbol{\tau^3} = (1, 0)$, $\boldsymbol{\tau^4} = (1, 1)$.
3) Regarding each information pool, the expected net utility $U_j^{\varepsilon}(x_j^{ip}|\boldsymbol{\tau})$ of account $j \in \{1, 2\}$ can be summarized as an payoff matrix

Table 1. Expected net utilities of the information pooling problems

		Account 2	
		$\tau_2 = 0$	$\tau_2 = 1$
Account 1	$\tau_1 = 0$	$(0.0149, 0.0149)$	$(0.0149, 0.0149)$
	$\tau_1 = 1$	$(0.0149, 0.0149)$	$(0.0065, 0.0065)$

4) Due to the same expected net utilities for both accounts with any of the information pools, the optimal market impact cost split ratio vector is $\boldsymbol{r^{\tau^i}} = (0.6414, 0.3586)$, $\forall i \in \{1, 2, 3, 4\}$, and the realized net utility vector is $U(\boldsymbol{x^{col}}|\boldsymbol{\tau}) = (0.0074401, 0.0074417)$. Moreover, the dissatisfaction indicators are

$$\begin{aligned} \boldsymbol{ds^{\tau^1}} = \boldsymbol{ds^{\tau^2}} = \boldsymbol{ds^{\tau^3}} &= (0.50066, 0.50055) \\ \boldsymbol{ds^{\tau^4}} &= (-0.14921, -0.14946) \end{aligned} \tag{11}$$

Remark 2. In this example, although the realized net utilities for both clients does not vary with the information pool, their vertical fairness is improved considerably by approximately 65%. It will help establish a stable manager-client relationship in practice.

If the resulting market impact cost is simply allocated in a pro rata fashion (O'Cinneide et al., 2006), the split ratio will be $\boldsymbol{r^{pr}} = (0.5883, 0.4117)$, and realized net utilities will be $U^{pr}(\boldsymbol{x^{col}}) = (0.0088, 0.0061)$. In this case, there is no information sharing between the clients, and the expected net utilities is corresponding to that with $\boldsymbol{\tau^1}$. Hence the dissatisfactions will be $\boldsymbol{ds^{pr}} = (0.4124, 0.5888)$, and client 2 with a small account is suffering almost 20% higher dissatisfaction compared to client 1. Moreover, our Pareto optimal information pooling solution $(0.0074401, 0.0074417)$ outperforms the Cournot-Nash equilibrium solution $(0.0647414, 0.0647414)$ for both accounts by sacrificing less than 0.03% horizontal fairness.

3.5. Discussion

This mechanism allows the manager to jointly optimize all clients' trading and split the market impact cost in a fair way. More precisely, the resulting market impact cost is allocated by minimizing the variance of dissatisfactions (vertical fairness) across all accounts (horizontal fairness). It produces Pareto optimal utilities while also keeps the satisfactions of all accounts at a similar level, complying with the SEC best execution rules.

From the clients' perspective, the information pooling game improves their perceived fairness from two aspects: first, it allows the clients to decide whether and to what extent to share their trading information, and their information pooling strategies will directly affect the market impact cost split ratio. A mechanism with involvement (voice) of investors is more likely to be regarded as fair. Second, it outperforms the pro-rata collusive solution in horizontal fairness, and overcomes the pitfall in Cournot-Nash equilibrium solution with a more tractable approach by introducing the expected net utility function.

4. Numerical Studies

In this section, two numerical studies are conducted to (1) compare the solutions in our mechanism to pro-rata collusive solution and Cournot-Nash equilibrium solution; (2) verify the information pooling effect on the realized net utilities and dissatisfaction indicators. In our numerical study, the return rate $\boldsymbol{\varpi}$ is randomly selected between -20% and 40%, and p is assumed to be 0.6 as in the example above.

4.1. Numerical Study on Three Solution Concepts

Suppose a manager is in charge of 2 portfolios $\boldsymbol{P} = \{1, 2\}$, and there are 50 assets $\boldsymbol{A} = \{1, 2, ..., 50\}$ available for investment. Account 1 has $w_1 = \$1M$ and account 2 has $w_2 = \$100M$ initially. Assume that the two clients reach a consensus on sharing their trading information regarding all assets, that is, $\tau_{ij} = 1, \forall i \in \boldsymbol{A}, \forall j \in \boldsymbol{P}$. This numerical study compares the performances of both accounts by collusive solution, Cournot-Nash equilibrium solution, and the information pooling solution proposed in this paper. The statistical results are summarized in Table 2 with properties of average expected/realized net utilities and dissatisfaction indicators reported.

Table 2. Account performances with collusive solution, Cournot-Nash equilibrium solution and information pooling solution

Property	Collusive		Cournot-Nash		Information Pooling	
	1	2	1	2	1	2
Avg. Expected Net Utility (%)	1.9487	1.2375	1.2814	1.1023	1.2814	1.1023
Avg. Realized Net Utility (%)	1.3495	1.1951	1.2814	1.1023	1.3857	1.1947
Dissatisfaction Indicator (%)	30.7487	3.4263	0.0000	0.0000	-8.1395	-8.3825

Remark 3. The statistical results above provide further evidence that

1) Managers and investors cannot afford to ignore the market impact cost. The mean of the return rate is set to be 10%, however, the net return after cost is only around 1%.
2) In collusive solution, the account with lower initial cash positions is hurt more. The actual market impact cost is significantly underestimated for clients without any information (Remark 1), which reduces their vertical perceived fairness considerably. As shown in Table 2, the realized net utility is approximately 30% lower than the prior expectation for account 1.
3) Cournot-Nash equilibrium solution is not Pareto optimal and violates the best execution rules, although it implements perfect horizontal fairness. For all accounts, both collusive and information pooling solutions bring about higher realized net utilities. (Figure 1).

4) Note that the expected net utility in our information pooling approach keeps consistent with that in Cournot-Nash equilibrium solution as all clients agree to disclose their trading information (Remark 1). Unlike the collusive solution, it rewards (or hurts) both accounts by approximately the same ratio. Compared to the Cournot-Nash equilibrium solution, although our approach sacrifices approx. 0.2% horizontal fairness, the Pareto optimal solution strictly improves the net utilities for both accounts by approx. 8%.

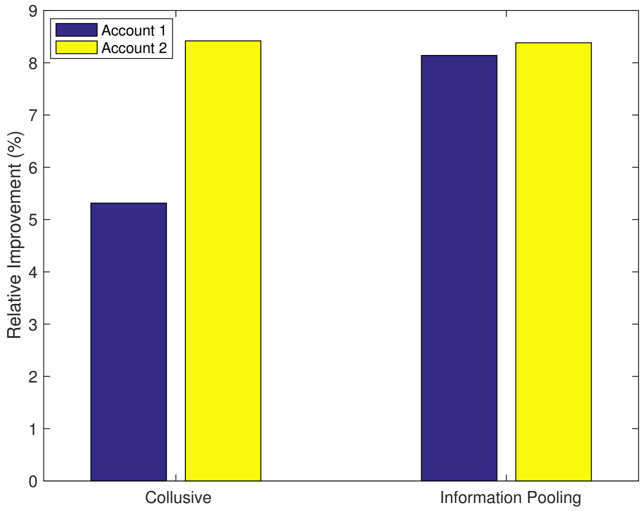

Fig. 1. Relative improvement of the realized net utilities with collusive and information pooling solutions compared to the Cournot-Nash equilibrium solution

4.2. Numerical Study on Information Pooling Game

Within our framework, this numerical study focuses on the effect of clients' information pooling strategies on the realized net utilities and perceived fairness. The setup is similar to the previous numerical study, but the manager is supposed to be in charge of 3 portfolios $\boldsymbol{P} = \{1, 2, 3\}$ with $w_1 = w_2 = \$1M$ and $w_3 = \$100M$ in initial cash positions. With respect to the 50 available assets, there are 2^{150} potential information pools and it exceeds the upper iteration limit of our program. Hence, four typical information pools will be compared, namely

1) Null information pool ($\tau_{ij} = 0, \forall i \in \boldsymbol{A}, \forall j \in \boldsymbol{P}$): all accounts decline to pool their trading information;
2) Partial information pool $(1-2)$ ($\tau_{i1} = \tau_{i2} = 1, \tau_{i3} = 0, \forall i \in \boldsymbol{A}$): accounts 1 and 2 with lower initial cash positions agree to pool their trading information;
3) Partial information pool $(1-3)$ ($\tau_{i1} = \tau_{i3} = 1, \tau_{i2} = 0, \forall i \in \boldsymbol{A}$): a small account and a large account consent to share their information, which is the same as partial information pool $(2-3)$.

4) Complete information pool ($\tau_{ij} = 1, \forall i \in \boldsymbol{A}, \forall j \in \boldsymbol{P}$): all accounts reach a consensus to disclose their trading information.

The average realized net utilities are summarized in Tables 3 and 4, and note that there are two pure strategy Nash equilibria $(\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \boldsymbol{\tau_3}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ or $(\mathbf{1}, \mathbf{1}, \mathbf{0})$. If we take the opposite of dissatisfaction indicators (satisfaction) as the payoff, the information pooling game can be represented in Tables 5 and 6, and the pure strategy Nash equilibria become $(\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \boldsymbol{\tau_3}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ or $(\mathbf{1}, \mathbf{1}, \mathbf{1})$. It indicates that although account 3 with higher initial cash position tends not to join the information pool if others do, it does perceive higher vertical satisfaction by acquiring more trading information.

Table 3. Avg. realized net utilities (%) if $\boldsymbol{\tau_3} = \mathbf{0}$

		Account 2	
		$\boldsymbol{\tau_2} = \mathbf{0}$	$\boldsymbol{\tau_2} = \mathbf{1}$
Account 1	$\boldsymbol{\tau_1} = \mathbf{0}$	$(\underline{1.3037}, \underline{1.3040}, \underline{1.1922})$	$(1.3037, \underline{1.3040}, 1.1922)$
	$\boldsymbol{\tau_1} = \mathbf{1}$	$(\underline{1.3037}, 1.3040, 1.1922)$	$(\underline{1.3086}, \underline{1.3088}, \underline{1.1921})$

Table 4. Avg. realized net utilities (%) if $\boldsymbol{\tau_3} = \mathbf{1}$

		Account 2	
		$\boldsymbol{\tau_2} = \mathbf{0}$	$\boldsymbol{\tau_2} = \mathbf{1}$
Account 1	$\boldsymbol{\tau_1} = \mathbf{0}$	$(1.3037, 1.3040, \underline{1.1922})$	$(1.2896, \underline{1.3113}, \underline{1.1923})$
	$\boldsymbol{\tau_1} = \mathbf{1}$	$(\underline{1.3113}, 1.2896, \underline{1.1923})$	$(\underline{1.3114}, \underline{1.3115}, 1.1920)$

Table 5. Opposite of avg. dissatisfaction indicators (%) if $\boldsymbol{\tau_3} = \mathbf{0}$

		Account 2	
		$\boldsymbol{\tau_2} = \mathbf{0}$	$\boldsymbol{\tau_2} = \mathbf{1}$
Account 1	$\boldsymbol{\tau_1} = \mathbf{0}$	$(\underline{-33.1573}, \underline{-33.1316}, \underline{-2.0217})$	$(-33.1573, \underline{-33.1316}, -2.0217)$
	$\boldsymbol{\tau_1} = \mathbf{1}$	$(\underline{-33.1573}, -33.1316, -2.0217)$	$(\underline{-30.2415}, \underline{-30.2457}, -1.9977)$

Table 6. Opposite of avg. dissatisfaction indicators (%) if $\boldsymbol{\tau_3} = \mathbf{1}$

		Account 2	
		$\boldsymbol{\tau_2} = \mathbf{0}$	$\boldsymbol{\tau_2} = \mathbf{1}$
Account 1	$\boldsymbol{\tau_1} = \mathbf{0}$	$(-33.1573, -33.1316, \underline{-2.0217})$	$(-33.8768, \underline{1.1962}, \underline{0.7095})$
	$\boldsymbol{\tau_1} = \mathbf{1}$	$(\underline{1.1962}, -33.8768, \underline{0.7095})$	$(\underline{3.3331}, \underline{3.3247}, \underline{2.3527})$

Remark 4. The comparative results shown in Figures 2 and 3 also suggest that

1) From the perspective of realized net utility, account 3 with higher initial cash position has less incentive to pool its trading information compared to the other two accounts, and complete information pool actually hurts its benefit compared to null information pool.
2) Even though some account chooses not to disclose its information, its realized net utility is made worse off by the existence of partial information pool, and a small account is hurt more.
3) From the perspective of fairness, information pooling process improves all participants' vertical fairness compared to the null situation, and has more impact on the small accounts as well.

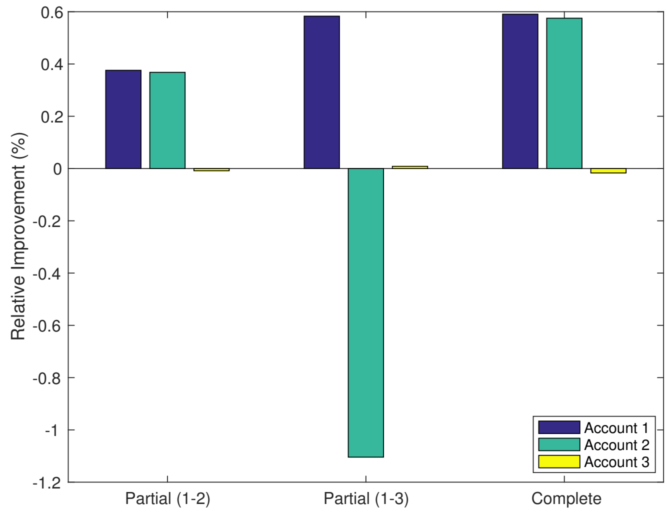

Fig. 2. Relative improvement of the realized net utilities with partial and complete information pools to that with null information pool

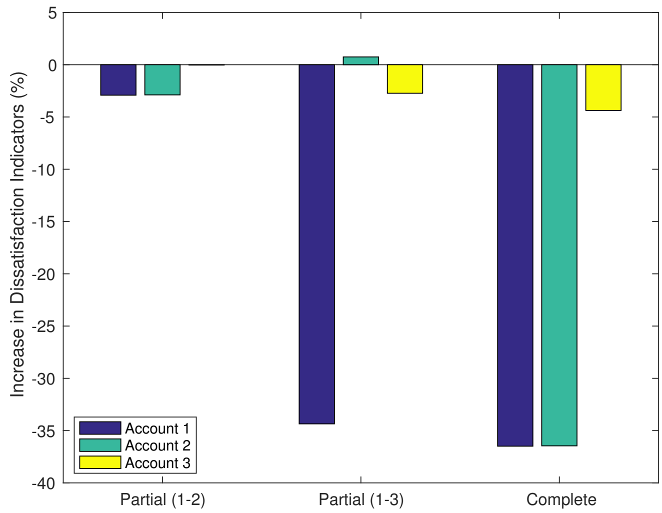

Fig. 3. Increase of dissatisfaction indicators with partial and complete information pools from that with null information pool

5. Conclusion

In this paper, an information pooling game for multi-portfolio optimization is introduced with incorporation of both horizontal and vertical perceived fairness from the clients' perspective. This novel mechanism invites the clients to decide whether and to what extent their trading information is shared with others, which directly affects the split ratio of resulting market impact cost in the collusive solution. It also allows the manager to jointly optimize multiple portfolios and split the market impact cost in a fair way by keeping the satisfactions of all accounts at a similar level.

The numerical study verifies that our approach outperforms the pro-rata collusive solution in fairness, and the Cournot-Nash equilibrium in Pareto optimality. It also suggests that small accounts are more sensitive to the information pooling process. For a more robust evidence, the existence of separate equilibria in multi-period information pooling game, as well as extensions with cross-asset effect still remain as our future work.

Acknowledgments. The author expresses her gratitude to S. Muto for useful discussions on the subjects.

Appendix

1. Proof of Lemma 1

Proof. With Definition 1, for account $k \in \boldsymbol{P}$, the first derivative of $c_i^k(\boldsymbol{x}|\boldsymbol{\tau})$ with respect to $x_{ik},\ \forall i \in \boldsymbol{A}$ can be derived as

$$\frac{\partial c_i^k(\boldsymbol{x}|\boldsymbol{\tau})}{\partial x_{ik}} = \begin{cases} p(x_{ik})^{p-1} & if\ \tau_{ik}=0 \\ p(\sum_{j\in\boldsymbol{P}}(\tau_{ij}x_{ij}))^{p-1} & if\ \tau_{ik}=1 \end{cases} \tag{12}$$

The second derivative with respect to x_{ik} can be derived as

$$\frac{\partial^2 c_i^k(\boldsymbol{x}|\boldsymbol{\tau})}{\partial x_{ik}^2} = \begin{cases} p(p-1)(x_{ik})^{p-2} & if\ \tau_{ik}=0 \\ p(p-1)(\sum_{j\in\boldsymbol{P}}(\tau_{ij}x_{ij}))^{p-2} & if\ \tau_{ik}=1 \end{cases} \tag{13}$$

We have ignored the cross-asset market impact, thus the second derivative with respect to $x_{hk},\ \forall h \neq i \in \boldsymbol{A}$ is 0 regardless of the value of τ_{ik}. As p is a rational number between 0.5 and 1, we have

$$\nabla c_i^k(\boldsymbol{x}|\boldsymbol{\tau}) > 0,\ \nabla^2 c_i^k(\boldsymbol{x}|\boldsymbol{\tau}) \leq 0 \tag{14}$$

Then for account k, the second derivative of $U_k^\varepsilon(\boldsymbol{x_k}|\boldsymbol{\tau})$ with respect to $x_{ik},\ \forall i \in \boldsymbol{A}$ can be represented as

$$\begin{aligned} &\frac{\partial^2 U_k^\varepsilon(\boldsymbol{x_k}|\boldsymbol{\tau})}{\partial x_{ik}^2} \\ &= \frac{\partial^2 u_k(\boldsymbol{x_k})}{\partial x_{ik}^2} - 2\frac{\partial c_i^k(\boldsymbol{x}|\boldsymbol{\tau})}{\partial x_{ik}} - x_{ik}\frac{\partial^2 c_i^k(\boldsymbol{x}|\boldsymbol{\tau})}{\partial x_{ik}^2} \\ &= \begin{cases} \frac{\partial^2 u_k(\boldsymbol{x_k})}{\partial x_{ik}^2} - (p^2+p)(x_{ik})^{p-1} < 0 & if\ \tau_{ik}=0 \\ \frac{\partial^2 u_k(\boldsymbol{x_k})}{\partial x_{ik}^2} - p(\sum_{j\in\boldsymbol{P}}(\tau_{ij}x_{ij}))^{p-2}(2\sum_{j\in\boldsymbol{P}}(\tau_{ij}x_{ij}) + (p-1)x_{ik}) < 0 & if\ \tau_{ik}=1 \end{cases} \end{aligned} \tag{15}$$

And the second derivative with respect to $x_{hk},\ \forall h \neq i \in \boldsymbol{A}$ is

$$\frac{\partial^2 U_k^\varepsilon(\boldsymbol{x_k}|\boldsymbol{\tau})}{\partial x_{ik}\partial x_{hk}} = \frac{\partial^2 u_k(\boldsymbol{x_k})}{\partial x_{ik}\partial x_{hk}} \leq 0 \tag{16}$$

For any account $j \in \boldsymbol{P}$, $U_j^\varepsilon(\boldsymbol{x_j}|\boldsymbol{\tau})$ is twice continuously differentiable and concave with respect to its own trade $\boldsymbol{x_j}$. Hence, $(\boldsymbol{x_j^{ip}})_{j\in\boldsymbol{P}}$ is an equilibrium of the information pooling problem if and only if $(\boldsymbol{x_j^{ip}})_{j\in\boldsymbol{P}}$ is a solution to the set of nonlinear complementary problems $\{NCP(\mathbb{R}_+^m, -\nabla U_j^\varepsilon(\boldsymbol{x_j}|\boldsymbol{\tau}))\}_{j\in\boldsymbol{P}}$ constrained by upper bounded trades of $\sum_{i\in\boldsymbol{A}} x_{ij} \le w_j$, $\forall j \in \boldsymbol{P}$. $\square$

2. Proof of Theory 2

Proof. For account $k \in \boldsymbol{P}$, assume the Hessian matrix of $U_k^\varepsilon(\boldsymbol{x_k}|\boldsymbol{\tau})$ to be

$$H = \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1m} \\ \vdots & \ddots & & & \vdots \\ a_{i1} & & a_{ii} & & a_{im} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & \cdots & a_{mi} & \cdots & a_{mm} \end{bmatrix}$$

Following the proof for Lemma 1, a_{ii}, $\forall i \in \boldsymbol{A}$ could be represented by equation (15), while a_{ih} and a_{hi}, $\forall h \neq i \in \boldsymbol{A}$ could be derived by equation (16). With $u_k(\boldsymbol{x_j}) = \boldsymbol{\varpi}^T\boldsymbol{x_k}$, it is very simple to show that

$$|a_{ii}| > \sum_{h\neq i} |a_{ih}|, \ |a_{ii}| > \sum_{h\neq i} |a_{hi}| \tag{17}$$

is satisfied for all $i \in \boldsymbol{A}$. Hence H is negatively strictly diagonally dominant. For any account $j \in \boldsymbol{P}$, the three conditions below are all satisfied

1) Expected net utility function $U_j^\varepsilon(\boldsymbol{x_j}|\boldsymbol{\tau})$ is twice continuously differentiable and concave with respect to $\boldsymbol{x_j}$;
2) Trades $\boldsymbol{x_j}$ is bounded;
3) $\nabla^2 U_j^\varepsilon(\boldsymbol{x_j}|\boldsymbol{\tau})$ has a negative strictly dominant diagonal for all $\boldsymbol{x_j} \ge \boldsymbol{0}$.

Based on K&M Theorem (Kolstad and Mathiesen, 1991), there exists a unique solution to $\{NCP(\mathbb{R}_+^m, -\nabla U_j^\varepsilon(\boldsymbol{x_j}|\boldsymbol{\tau}))\}_{j\in\boldsymbol{P}}$. $\square$

References

Almgren, R., C. Thum, E. Hauptmann and H. Li (2005). *Equity market impact.* Risk, 57–62.

Bies, R. J., T. M. Tripp and M. A. Neale (1993). *Procedural fairness and profit seeking: the perceived legitimacy of market exploitation.* Journal of Behavior Decision Making, **6**, 243–256.

Fabozzi, F., S. Focardi and P. Kolm (2010). *Quantitative Equity Investing: Techniques and Strategies.* John Wiley & Sons: Hoboken, NJ.

Iancu, D. A. and N. Trichakis (2014). *Fairness and efficiency in multiportfolio optimization.* Operations Research, **62(6)**, 1283–1301.

Kahneman, D., J. L. Knetsch and R. H. Thaler (1986). *Fairness and the assumptions of economics.* Journal of Business, **59(4)**, 285–300.

Kolstad, C. D. and L. Mathiesen (1991). *Computing Cournot-Nash Equilibria.* Operations Research, **39**, 739–748.

Markowitz, H. (1952). *Portfolio selection.* J. Finance, **7(1)**, 77–91.

Mas-Colell, A. (1984). *On a theorem of Schmeidler.* J. Math. Econom. , **13**, 201–206.

O'Cinneide, C., B. Scherer and X. Xu (2006). *Pooling trades in a quantitative investment process.* J. Portfolio Management, **32(4)**, 33–43.

PlexusGroup (2002). *Sneaking an elephant across a putting green: a transaction case study.* Commentary 70.

Savelsbergh, M. W. P., R. A. Stubbs and D. Vandenbussche (2010). *Multiportfolio optimization: a natural next step.* Handbook of Portfolio Construction, 565–581.

Securities and Exchange Commission (2011). *General information on the regulation of investment advisers.* Retrieved on February 15, 2017, `https://www.sec.gov/divisions/investment/iaregulation/memoia.htm`.