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Cost Optimization for the Transport Network of Yakutia

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Abstract The paper studies game-theoretic approach to the problem of reducing costs of agricultural products transportation on the transport network roads of the Sakha (Yakutia) Republic. Also were proposed a mathematical formulation of problem of players cooperation as the problem of reducing costs for the transport network in the form of cooperative game with characteristic function, as well as a cooperative game with a coalition structure. The solution of such cooperative games, i.e. the optimal cost distribution between players was found in the form of Shapley value.

Keywords: cooperative game, characteristic function, Shapley value.

1. Introduction

The economic development of the Russian Far East and Siberia depends on the transport network. The transport network of Sakha (Yakutia) Republic has special features such as that hard-surface roads connect only 7,3% of all settlements of republic. At the same time 30% of all roads are hard-surface roads and the remaining 70% are ice roads.

The game-theoretical approach to the problem of cost optimization of agricultural products transportation on the transport network of the Sakha (Yakutia) Republic is considered in this paper. Also is proposed a mathematical formulation of problem of players cooperation as the problem of reducing costs for the transport network in the form of cooperative game with characteristic function, as well as a cooperative game with a coalition structure.

The application of cooperative game theory for similar problems were reviewed in (Ergun et al., 2007; Krajewska et al., 2008; Khmelnitskaya and Yanovskaya, 2007). Transport network can be represented as a finite graph, defined by a finite set of vertices and edges. The vertices of this graph are settlements, and the edges are roads with a different type of road surfaces. The goal of each player is to transport products to Yakutsk, and the player's strategy is to choose the way with minimal transportation costs. Transport network is considered in summer and winter periods.

The cooperative game theory helps to construct a mathematical model of economic problems of transporting products (Von Neumann and Morgenstern, 1944). Such problems required minimizing the total costs of transporting products, for example by creating various coalitions of participants using this road network.

Games with a coalition structure were studied in papers that described cooperative solutions in the form of the Owen value and the Aumann-Dreze value (Owen, 1971; Aumann and Dreze, 1974). In this paper were proposed a mathematical formulation of the problem of cooperation between producers of agricultural products in transporting it and the construction of a cooperative game in the form of a characteristic function in the problem of transporting manufactured products. The cooperative theory of games allows to investigate the possibilities of producers of agricultural products in order to reduce the costs of its transportation. As a principle of optimality of cooperative game we took the Shapley value (Shapley, 1953). The Shapley value is an n-dimensional vector in which the component i is a cost of the player i. Thus, the Shapley value determines the optimal distribution of total costs incurred by the maximum coalition (coalition of all players) as a result of the cooperative effect. A mathematical model in the form of cooperative games in similar problems was investigated in the papers (Zakharov and Shchegryaev, 2012; Shchegryaev and Zakharov, 2014).

Classical cooperative model assumes that it is advantageous for players to unite into a maximal coalition in order to minimize total costs due to the subadditivity of characteristic function. Therefore, the problem is to find a sharing of the minimum costs.

So, it is assumed that there are several producers of agricultural products located in the ulus centers, which are connected to the city of Yakutsk road network. Each manufacturer has a certain own resource, designed to transport the products from the ulus center in Yakutsk. In order to reduce the cost of transporting agricultural products, local producers can be combined into different coalitions. Therefore, in each coalition, there may be a redistribution of costs between the participants of this coalition, and as a consequence, a change in the way (strategy) of transporting the products of each producer in comparison with the ways planned without taking into account possible cooperation. Then, firstly, it is required to find the optimal way for each player of the coalition and determine its total costs, and secondly, to calculate the values of characteristic function of the corresponding cooperative game.

On the basis of the foregoing, the task is to study this optimization problem in the form of a cooperative game in order to effectively divide the total costs between producers of products.

Let $S \subseteq N$ be some coalition of players. The minimum aggregate costs $v_h(S)$ of the S coalition are the sum of the minimum total costs of each player i of the coalition S. The cost of the transition to the common path is included in this amount once, see (Karpov and Petrosyan, 2012). This can be interpreted in such a way that the coalition S to overcome the common path carries costs proportional to the length of this path. The cost of passing the ferry line of the coalition S is determined by the carrying capacity of the vehicle. For example, a player with the longest minimum path among all participants in the game offers services for transporting products to other players to overcome the common path, assuming that transportation costs are divided among the players of the coalition S in accordance with the Shapley value.

The following is a review of the literature on which the research is based in this paper.

In the paper (Karpov and Petrosyan, 2012) cooperative solutions in communication networks are studied. The algorithm for constructing the optimal allocation of costs among players constructed in this work allows us to find rational solutions of cooperative games on communication networks. The study of optimization problem in the form of a cooperative game makes it possible to effectively divide costs among players in the form of the Shapley value.

In the paper (Zakharov and Shchegryaev, 2012) the problem of optimizing transportation costs for carriers' cooperation on networks is considered. The developed algorithm of the coalition induction of the construction of the characteristic function in the cooperation of transport companies, which ensures the performance of the property of subadditivity of the characteristic function, allowing us to find the optimal distribution of costs. As a solution, the Shapley value is considered. As a result of the proposed algorithm, effective solutions of the corresponding cooperative game-effective routes for cargo transportation-are constructed to obtain the characteristic function. Then this algorithm allows to determine the distribution of optimal costs between players in the form of the Shapley value.

2. The game-theoretic approach to the problem of cost reduction for the transport network of Yakutia

2.1. Mathematical model of the problem of cooperation of players with the purpose of reducing transportation costs

Let $G = (X, R)$ be the graph, where X is the set of vertices x_i, $i = 1, 2, ..., n$, representing settlements of the republic, R is the set of edges $r_{ij} = (x_i, x_j)$, $j = 1, 2, ..., n$, that represents the model of the roads. The edges have weighted numerical characteristics determined by the type of road surface. The weight characteristics p_{ij} satisfy the equality $p_{ij} = p_{ji}$. Thus, this weight characteristics form a symmetric matrix $P = (p_{ij})$. Since every edge of the graph G has a weight, this graph is a network. A conventional graph in this sense is a network whose weight p_{ij} of each edge (x_i, x_j) is equal to one.

A non-negative symmetric real function $S_{ij} = s(r_{ij})$ is defined on the set of edges R. The value of this function determines costs that are associated with the passing from vertex x_i to vertex x_j on the edge r_{ij}.

It is clear that $s_{ij} = s_{ji}$ and $s_{ii} = 0$. In addition to the function $s_{ij} = s(r_{ij})$, we also introduce function $L_{ij} = l(r_{ij})$ as follows

$$L_{ij} = p_{ij} s_{ij},$$

where p_{ij} is coefficient of road surfaces or edge weight characteristic.

Now let us give some theoretical basis for the formulation and solution of problem in the form of cooperative game.

Network state $G = (X, R)$ is an n-dimensional vector $z = (z^1, z^2 ..., z^n)$ such that its every component corresponds to the vertex of graph G in which located player i (Karpov and Petrosyan, 2012). We denote set of states of network G by Ω. By *path* $L(r_{ij})$ from vertex x_i to vertex x_j of network G is meant any finite sequence of edges from the set R that connect the vertex x_i to the vertex x_j.

Now consider n-person game on network $G = (X, U)$, where $N = 1, 2, ..., n$ is the set of players, and z_0 and z_l is the initial and final states of the network G. We denote this game by $\Gamma = (G, N, z_0, z_l)$.

By *strategy* h^i of player i is meant any path connecting its initial position z_0^i with finite position $z_l^i, i = 1, 2, ..., n$. By $H(\{i\})$ denoted the set of all strategies of player i, and by $H(N)$ denoted the set of all possible situations.

Next we denote $U(H)$ as the set of edges in the situation $h = (h^1, h^2, ..., h^n)$. A feature of this set is that it contains only different edges.

We also introduce the concept of *total costs* in the Γ game corresponding to the situation of h, as follows (Karpov and Petrosyan, 2012):

$$l(h) = \sum_{r_{ij} \in U(h)} l(r_{ij}). \tag{1}$$

Trajectory in a network G is defined as a sequence of network states $(z_1, z_2, ..., z_m)$, where $z_k = (z_k^1, z_k^2, ..., z_k^n)$, $k = 1, 2, ..., m$. Each network state z_k defines n vertices in which the players are located, i.e. each trajectory corresponds to n player strategies and a certain situation h. By $P(z_1, z_m)$ denoted the trajectory of passing from the state z_1 to the state z_m.

The optimal trajectory in the network G is the trajectory $P^*(z_1, z_m)$ that corresponds to the situation h^* which minimizes the total costs $l(h^*)$ in the game Γ that is:

$$l(h^*) = \min l(h). \tag{2}$$

Consider the following problem: find the optimal trajectory in n-person game $\Gamma = (G, N, z_0, z_l)$ on the network with the initial state z_0 and the final state z_l. The solution of this problem can be found using dynamic optimization, precisely on the basis of Bellman's optimality principle (Bellman, 1960). This optimality principle allows us to find the recurrence functional Bellman equation.

Now define the characteristic function $v(S)$ in a recurrent way (Zakharov and Shchegryaev, 2012; Shchegryaev and Zakharov, 2014):

$$v(S) = \min\{\min_{Q \subset S}(v(Q) + v(S \setminus Q)); v_h(S)\}. \tag{3}$$

The representation of the characteristic function in the form (3) can be interpreted as the coalition S minimum guaranteed cost.

The cost $v_h(S)$ of the coalition S is composed of two components: the direct costs $l_h(S)$ for transporting agricultural products along the path $U(h_s) \setminus U_0(h_s)$, where $U_0(h_s)$ is the set of common edges in the path $h^i \in S$ and the cost $a_h(S)$ of the coalition S to overcome the path $U_0(h_s)$. Hence follow equality:

$$v_h(S) = l_h(S) + a_h(S),$$

where

$$l_h(S) = \alpha \sum_{r_{ij} \in U(h_s) \setminus U_0(h_s)} l(r_{ij}),$$

$$a_h(S) = \alpha \sum_{r_{ij} \in U_0(h_s)} l(r_{ij})$$

and α is the cost of transporting agricultural products per unit of path. Assuming that $\alpha = 10$ from the calculation that cost of 1 liter of gasoline is 50 rubles, and crossing 1 kilometer of the path it will take 10 rubles. Maximum costs for the ferry crossing by car Nizhny Bestyakh — Yakutsk depend on the weight of the car: up to

1 ton (transportation of agricultural products by one manufacturer) - - 340 rubles, up to 2 tons (transportation by two manufacturers) – 610 rubles, up to 3 tons (transportation by three manufacturers) – 935 rubles, up to 4 tons (transportation by four manufacturers) – 1275 rubles. The costs of passing the ferry crossing for the coalition S are determined by the carrying capacity of the vehicle and are presented in the table 1.

Table 1. Costs of the coalition to overcome the ferry crossing

s	Costs of coalition S(in rubles)
1	340
2	610
3	935
4	1275

For the four person game Γ we have the following algorithm for calculating the values of the characteristic function:

Step 1. Find the value of the characteristic function for singleton coalitions

$$v(\{i\}) = v_h(\{i\}),$$

where $h \in H(\{i\})$.

Step 2. Compute the value of the characteristic function for two-element coalitions

$$v(\{i,j\}) = min\{v(\{i\}) + v(\{j\}); v_h(\{i,j\})\},$$

where $h \in H(\{i,j\})$, $i \neq j, i,\ j = 1,2,3,4$.

Step 3. Find the value of the characteristic function for three-element coalitions

$$v(\{i,j,k\}) = min\{v(\{i,j\}) + v(\{k\}); v_h(\{i,j,k\})\},$$

where $h \in H(\{i,j,k\})$, $i \neq j,\ j \neq k,\ k \neq i,\ i,j,k = 1,2,3,4$.

Step 4. Calculate the value of the characteristic function for the maximum coalition

$$v(N) = min\{v(\{i,j,k\}) + v(\{l\}); v(\{i,j\}) + v(\{k,l\}); v_h(\{i,j,k,l\})\},$$

where $h \in H(\{i,j,k,l\})$, $i \neq j,\ j \neq k,\ k \neq l,\ l \neq i,\ i,j,k,l = 1,2,3,4$.

2.2. The problem of reducing cost for transport network of Yakutia as a cooperative game in the form of a characteristic function

Consider cost reducing optimization problem of transporting agricultural products on transport network of the Sakha (Yakutia) Republic.

There are several producers of agricultural products located in the ulus centers: village Amga (Amginskiy ulus), village Churapcha (Churapchinsky ulus), village Borogontsy (Ust-Aldansky ulus) and village Pavlovsk (Megino-Kangalassky ulus). Ulus centers are connected with the city of Yakutsk by transport network and depicted as a graph. Taking into account the coefficient of road surface, which is equal to 0.1 for roads with smooth surface; 0,25 for roads with a hard surface; 0,5 for ice roads; 0,75 for dirt roads and 1 for ferry line (Evtyukov and Evtyukov, 2013), the graph of transport network is presented in Fig. 1. This graph represents to the summer state of transport network.

Each manufacturer has a certain resource for transporting its products from an ulus center to Yakutsk. The goal of each player is to transport agricultural products with minimal costs, i.e., finding the way in which transportation costs will be minimal.

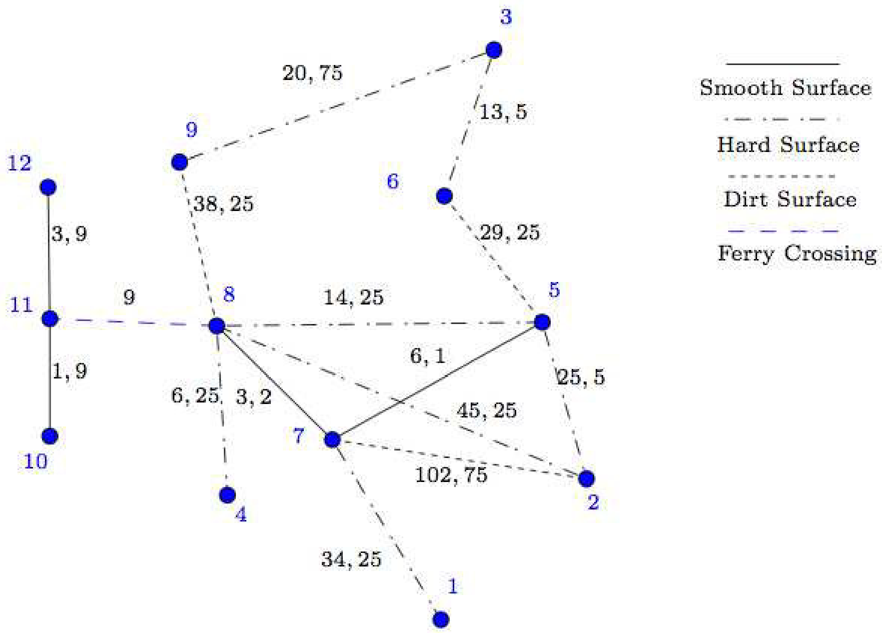

Fig. 1. The graph of summer state of highway network

The initial state of the network is the state $z_0 = (1, 2, 3, 4)$, and the final state is $z_l = (11, 11, 11, 11)$, that is, for example, the first producer of products must overcome the path from the vertex 2 to the vertex 11.

Based on the algorithm described in work (Karpov and Petrosyan, 2012) we find the optimal routes for the producers:

$$h^{*1} = \{(1, 7), (7, 8), (8, 11)\};$$
$$h^{*2} = \{(2, 5), (5, 7), (7, 8), (8, 11)\};$$
$$h^{*3} = \{(3, 6), (6, 5), (5, 7), (8, 11)\};$$
$$h^{*4} = \{(4, 8), (8, 11)\}.$$

Taking into account the ferry transportation tariffs we find the following minimal costs for the each manufacturer:

$$(l(h^{*1}),\, l(h^{*2}),\, l(h^{*3}),\, l(h^{*4})) = (714.5;\, 688;\, 860.5;\, 402.5)$$

and, additionally

$$l(h^*) = \sum_{i\in N} l(h^{*i}) = 2665.5.$$

Now let us consider cost reducing optimization problem which differs from the previous one. This time graph corresponds to road network in winter. The transition from the road network in summer period to summer period corresponds in the context of the work (Butenko, 2015) to external influences — "shocks". We will find out how the optimal ways of the producers of products change during the formation of winter crops.

Taking into account the coefficient of road surface the graph of transport network is presented in fig. 2.

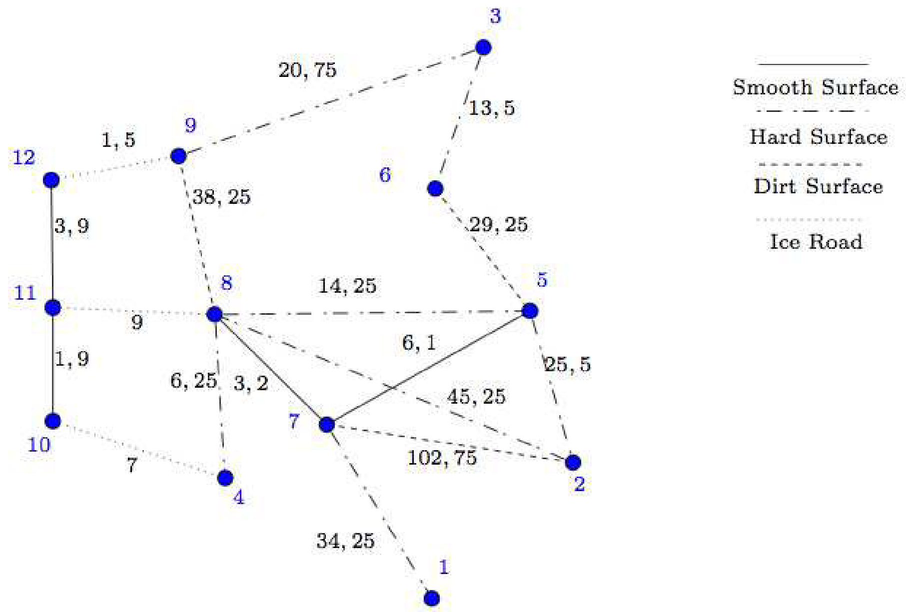

Fig. 2. The graph of winter state of highway network

Thus, each of manufacturers carries following optimal costs:

$$(l(h^{*1}), l(h^{*2}), l(h^{*3}), l(h^{*4})) = (464,5; 438; 261,5; 89)$$

and, additionally

$$l(h^*) = \sum_{i\in N} l(h^{*i}) = 1253.$$

Comparing calculated optimal costs vectors of two optimization problems we can see that graph changing, based on ice roads formation implies decrease of each agricultural manufacturers optimal costs.

The initial state of the network is the state $z_0 = (1,\ 2,\ 3,\ 4)$, and the final state is $z_l = (11,\ 11,\ 11,\ 11)$, that is, for example, the first producer must follow the path from the vertex 2 to the vertex 11.

According to the algorithm described in work (Karpov and Petrosyan, 2012) were found the optimal ways for the transportation of the products:

$$h^{*1} = \{(1,7),(7,8),(8,11)\};$$
$$h^{*2} = \{(2,5),(5,7),(7,8),(8,11)\};$$
$$h^{*3} = \{(3,9),(9,12),(12,11)\};$$
$$h^{*4} = \{(4,10),(10,11)\}.$$

Cooperative game of n persons is the game $\Gamma = (N, v)$, where $N = \{1, 2, ..., n\}$ is the set of players, and v is the characteristic function defined on the set coalition.

Let us now look at a 4-person cooperative game $\Gamma_1^* = (N, v)$ in the form of the characteristic function on the network G. The need for coalition formation can be justified as follows. Let $S \subseteq N$ be some coalition of players. The minimum cumulative costs $v_h(S)$ of the coalition S are the sum of the minimum total costs of each player i of the coalition S. The cost of the transition by the common path is included in this amount only once regardless the number of players using this path (Karpov and Petrosyan, 2012).

For example, the player with the longest minimum path among all participants in the game offers services to other players to cross the common path, assuming that transport costs are divided among the players of the S coalition in accordance with the Shapley value.

We find the value of the characteristic function for all possible coalitions and write its values as table 2.

Table 2. Values of the characteristic functions for the cooperative game Γ_1^*

S	$v(S)$	S	$v(S)$
{1}	714.5	{2,4}	1020.5
{2}	688	{3,4}	1193
{3}	860.5	{1,2,3}	2053
{4}	402.5	{1,2,4}	1688
{1,2}	1300.5	{1,3,4}	1860.5
{1,3}	1473	{2,3,4}	1773
{1,4}	1047	N	2432.5
{2,3}	1385.5		

Now let us consider the distribution of costs between players. As optimality principle, we choose the Shapley value, each component of which is the spending of the corresponding player:

$$Sh_i(v) = \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{i\}),\ s = |S|,\ i = 1,2,3,4. \quad (4)$$

Computing the components of the Shapley value, according to this rule, we obtain

$$Sh^*(v) = (666.167;\ 609.167;\ 781.667;\ 375.5)$$

and at the same time completing its property of efficiency

$$\sum_{i \in N} Sh_i^*(v) = 2432.5.$$

The results for two non-cooperative games Γ_1 and Γ_2, corresponding to the summer and winter seasons along with the cooperative variant of the first game referred to as Γ_1^* are presented in Tables 3 and 4.

Table 3. Non-cooperative games Γ_1 and Γ_2

Player	Optimal path in Γ_1	Costs	Optimal path in Γ_2	Costs
1	(1,4),(4,8)(8,11)	714,5	(1,4),(4,8)(8,11)	464,5
2	(2,5),(5,8)(8,11)	737,5	(2,5),(5,8)(8,11)	487,5
3	(3,9),(9,8)(8,11)	930	(3,9),(9,12)(12,11)	680
4	(7,8)(8,11)	402,5	(7,10)(10,11)	152,5

Table 4. Cooperative game Γ_1^*

Player	Optimal path in Γ_1^*	Components of the Shapley value
1	((1,4),(4,8)(8,11)	573,25
2	(2,5),(5,8)(8,11)	596,25
3	(3,9),(9,8)(8,11)	788,75
4	(7,8)(8,11)	261,25

We can observe that in the cooperative version of the game the players bear smaller costs when the game takes place in the summer period. In the winter period, however, the transportation costs turn out to be smaller than in the summer period and there is no opportunity for cooperation.

3. Conclusion

In this paper was studied game-theoretic approach to the problem of reducing costs of agricultural products transportation on the transport network of the Sakha (Yakutia) Republic. The game-theoretic approach allows producers of agricultural products, acting together bear the minimum costs for its transportation.

The mathematical model of the problem of reducing costs for transportation of products studied in the work can be used to solve similar economic problems associated with transport costs. The practical application of this kind of mathematical models in such problems of transporting products will help to avoid unnecessary spending in transportation of products.

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