Contributions to Game Theory and Management, XI, 249–259

Schelling Point as a Refinement of Nash Equilibrium

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Abstract This article proposes using Schelling point as a refinement of Nash equilibrium. It also introduces an algorithm way to locate Schelling points in games. The paper shows that in pure coordination games and games with significant proportion of coordination (and certain proportion of conflict), the algorithm proposed could locate the Schelling points. The existence of a Schelling point means that the proportion of common interests versus conflict of interests has crossed a certain threshold and therefore a certain form of cooperation or coexistence is possible. Besides, using Schelling point as a refinement of Nash equilibrium narrows down the set of equilibriums to only stable equilibriums as unstable equilibriums could not be Schelling points. Finally, the paper shows that the proposed approach could solve a larger set of games than the current approach based on Nash equilibrium. For instance, it would be very difficult to solve multiple sided incomplete information games under the current approach based on Nash equilibrium while the proposed approach could readily solve multiple sided incomplete information games.

1. Introduction

Von Neumann and Morgenstern (1944) is the first major work in games theory. It established the field of cooperative games theory. Then came John Nash (1951)'s non-cooperative games theory. Non-cooperative games theory is now the prevalent form of games theory used to analyze topics of interest in social sciences and management sciences. The basic solution concept of noncooperative games theory is Nash equilibrium. But, there are many problems with the solution concept of Nash equilibrium, such as the existence of too many equilibrium including many which hardly make any sense and, the problem of interpreting the mixed strategy equilibrium. These problems prompted many debates and critics including Nobel laureate Aumann (1985)'s questioning "what is games theory trying to achieve?" Consequently, there are many efforts to refine the solution concept of the non-cooperative games theory as well as efforts to search for alternative equilibrium concept.

Non-cooperative games theory is however not totally non cooperative as the name implies, as pointed out by Schelling (1960). There are both elements of cooperation and conflict in nonzero sum games. Given the coexistence of both elements of conflicts and cooperation, the key to the solutions to these games is on how to achieve cooperation or coordination in the face of the existence of conflict, argued Schelling (1960). The existence of elements of conflicts means that communication might not be easy neither could the other party to be trusted. Therefore, the concept of focal point proposed by Schelling is useful and important and it could lead to coordination of the parties for better mutual benefits. A focal point is a point of convergence of expectations or beliefs without communication. Focal point is also known as Schelling point given Schelling (1960)'s contribution.

There are many researches on the concept of Schelling point.[1] [2] This paper proposes an algorithm to locate the Schelling point in games and use the Schelling point as a refinement for Nash equilibriums. The method proposed could find the Schelling point in non-zero sum games and use it as refinement for Nash equilibriums (and the many refinements of Nash equilibrium) for nonzero sum games, including games with incomplete information and sequential moves. The algorithm could locate the Schelling point even in situations where the Schelling point is not Pareto dominant.

A Schelling point is where expectations or conjectures converge. Therefore, how conjectures are formed and updated affects the determination of Schelling points. The algorithm proposed here has two key features. First is that it starts with the conjecture or assumption that all possible strategies are equally likely. The first order conjectures each has a diffuse uniform probability distribution function which gives equal probability to all possible actions. This is to avoid convergence by assumption.

Second is that it then uses iterative conjectures to improve upon the first order conjectures until a convergence is reached, if there is a convergence. The updating of conjectures is based upon game theoretic reasoning. The optimal strategies of the players given the first order conjectures formed the second order conjectures, and the optimal strategies of the players given the second round of conjectures formed the third order conjectures, and so forth. Of course, statistical reasoning, especially updating by Bayes rule, are involved in the revising of information and conjectures.

Section 2 studies the simplest two player two actions simultaneous games of complete information. It shows that a Schelling point exists if the game has strong elements of coordination or common interest, that is, if the game has a best response equivalent identical interest game. Section 3 illustrates the relationship between common interest and Schelling point for a sequential incomplete information game. Section 4 solves for the Schelling point of a sequential double sided incomplete information game to illustrate the point that Schelling point could solve more games that the current approaches based on Nash equilibrium. Section 5 concludes the discussions.

2. The algorithm

This section introduces the algorithm that will select the Schelling point in the simplest context of two players and two actions complete information simultaneous games. This section also illustrate on the relationship between common interest in games and the existence of a Schelling point. Specifically, this section shows that for two players and two actions complete information simultaneous games, if a game has a best response equivalent identical interest game, then at least a pure strategy Nash equilibrium exists and therefore the game has a Schelling point. A Schelling point is by its very nature a stable Nash equilibrium since it is where conjectures or expectations converge. A mixed strategy Nash equilibrium is not a stable Nash equilibrium and therefore could not be a Schelling point. A best-response equivalent game is a transformation of a game whereby the payoff matrix of the original game is

[1] Examples are Sugden (1995), Colman (1997) and Crawford, Gneezy, and Rottenstreich (2008).

[2] Refer to Teng (2013) for an earlier treatment of convergence of conjectures and Bayesian updating in the process.

transformed yet the reaction functions are preserved so that the strategic nature of the game is unchanged.[3] An identical interest game has the special feature that the payoffs of the players are the same. In an identical interest game, there is therefore at least a natural focal point or Schelling point: the strategy that yields the highest payoff.[4] In contrast, zero sum games are strictly competitive games or pure conflict games where there are no common interests at all. A zero sum game has the special feature that the payoffs of one of the players are exactly the negative of the other player. A zero sum game therefore naturally has no focal point or Schelling point. A zero sum game could not be represented by a best response equivalent identical interest game and has only a mixed strategy Nash equilibrium. A mixed strategy Nash equilibrium is unstable. Therefore, it could be not a Schelling point arrived by the convergence of conjectures approach.

First, we look at games with no pure strategy Nash equilibrium and only a mixed strategy Nash equilibrium. Table 1 below gives the normal form representation of a two player, two action complete information simultaneous game with only a mixed strategy Nash equilibrium and no pure strategy Nash equilibrium.

Table 1.

1\2	L (y)	R (1-y)
L (x)	1, 10	0, 30
R (1-x)	0, 50	5, 20

Table 2. No Convergence of Conjectures

Order of Conjectures	x	y
1	1/2	1/2
2	0	1
3	1	1
4	1	0
5	0	0
6	0	1

The 6^{th} order conjectures are the same as the 2^{nd} conjectures and the conjectures do not converge. Table 3 gives the normal form representation of a best response equivalent zero sum game of the game in Table 1. Proposition 1 generalizes the

Table 3.

1\2	L (y)	R (1-y)
L (x)	2, -2	-10, 10
R (1-x)	-3, 3	15, -15

insight from the above example.

Proposition 1. *A two player two action simultaneous move complete information game with only a unique mixed strategy Nash equilibrium and no pure strategy Nash*

[3] Morris and Ui (2004).

[4] Carlsson and van Damme (1993).

equilibrium could be represented by a best-response equivalent zero sum game and could not be represented by a best-response equivalent identical interest game.

Proof. For the first part, suppose the game is represented by the following best-response equivalent zero sum game: where $a, b, c, d > 0$.

Table 4.

1\2	L (y)	R (1-y)
L (x)	-a, a	d, -d
R (1-x)	c, -c	-b, b

Note that having no pure strategy Nash equilibrium requires that $c > -a$, $b > -c$, $d > -b$, $a > -d$ which could be satisfied given the assumptions and involves no contradiction.

For the second part, suppose that there is an identical interest game that has no pure strategy Nash equilibrium and has only a unique mixed strategy Nash equilibrium. Without loss of generality, consider the following identical interest game: Assume that $c > a, b > d, a > b$ and $d > c$ for the absence of pure strategy equilibrium. From the aforementioned inequalities we have $a > b > d > c > a$. Yet, the above is self-contradictory. Q. E. D.

Table 5.

1\2	E(y)	R(1-y)
M(w)	a,a	d,d
A (1-w)	c,c	b,b

Next, we look at games with two pure strategy Nash equilibriums and a mixed strategy Nash equilibriums. These games are at least partially coordination games as the core of the problem is often about how to coordinate the actions of the players such that one of the pure strategy Nash equilibrium is selected. All games with two pure strategy Nash equilibriums and a mixed strategy Nash equilibrium could be transformed into identical interest games.

Table 6 below gives the normal form representation of a game with two pure strategy Nash equilibriums and a mixed strategy Nash equilibrium. Table 7 below gives the best response equivalent identical interest game of the game in Table 6. In Table 7, Player 1 playing A and Playing 2 playing E is the Pareto dominant

Table 6.

1\2	E (y)	R (1-y)
M (x)	0, -2	7, 0
A (1-x)	4, 2	6, 0

Table 7.

1\2	E (y)	R (1-y)
M (w)	0, 0	5, 5
A (1-w)	8, 8	3, 3

outcome and therefore the natural Schelling point.

Proposition 2 generalizes the insight.

Proposition 2. *A two player two action simultaneous move complete information game with a mixed strategy Nash equilibrium and two pure strategy Nash equilibriums could be represented by a best-response equivalent identical interest game and could not be represented by a best-response equivalent zero sum game.*

Proof. For the first part, suppose the game is represented by the following best-response equivalent identical interest game: where $a, b, c, d > 0$. WLOG assume

Table 8.

1\2	U	D
U	a, a	c, c
D	b, b	d, d

that the two pure strategy Nash equilibriums are (U, U) and (D, D). The above assumption requires that a>=c, a>=b, d>=c, d>=b which could be satisfied given the assumptions and involves no contradiction.

For the second part, suppose that there is a zero sum game that has two pure strategy Nash equilibriums and a mixed strategy Nash equilibrium. Without loss of generality, consider the following zero sum game: where $a, b, c, d > 0$.

Table 9.

1\2	L	R
L	-a, a	c, -c
R	b, -b	-d, d

Without loss of generality assume that (L, L) and (R, R) are the pure strategy Nash equilibriums.

That requires $- - a >= b, -d >= c, a >= -c$ and $d >= -b$.

So, from the above inequalities we have $-a >= b >= -d >= c >= -a$.

Note that except for the case of $-a = b = -d = c = -a$ which contradicts the assumption that $a, b, c, d > 0$, the above inequalities are contradictory. Q. E. D.

Next, we consider games with only one pure strategy Nash Equilibrium. Such a game has both pure conflict best response equivalent games and pure coordination best response equivalent games. A good example is the famous prisoners' dilemma game.

Proposition 3. *If there is only one pure strategy Nash Equilibrium, then the game has both pure conflict best response equivalent games and pure coordination best response equivalent games.*

Proof. Consider the identical interest game best-response equivalent first: where $a, b, c, d > 0$.

(D, D) is the unique pure strategy Nash equilibrium if $b > a, c > a, d > c, d > b$. The above lead to $d > b, c > a$ which has no contradictions.

Now consider the zero sum game best-response equivalent:

(D, D) is the unique pure strategy Nash equilibrium if $b > -a$, $-c > a$, $-d > c$, $d > -b$. The above lead to $b > -a$, $-d > c$ which has no contradictions. Q. E. D.

Table 10.

1\2	Cooperate	Defect
Cooperate	a,a	c,c
Defect	b,b	d,d

Table 11.

1\2	Cooperate	Defect
Cooperate	-a,a	c,-c
Defect	b,-b	-d,d

Summing up proposition 1 to 3, we learn that a two player two action simultaneous move complete information game with at least a pure strategy Nash equilibrium could be represented by a best-response equivalent identical interest game. So existence of a pure strategy Nash equilibrium means that the game has the nature (at least partially) of a coordination game (or identical interest game). Then Schelling point is a good solution concept for such games.

Continue with our example of Table 6. We now solve it by looking for the Schelling point through convergence of conjectures. Let the probability that player 1 plays M be x and the probability that player 2 plays E be y. The first order conjectures are $Pr(x) = 1/2$ and $Pr(y) = 1/2$.

Table 12.

Order of Conjectures	x	y
1	1/2	1/2
2	0	1/2
3	0	1
4	0	1

Note that the third and fourth order conjectures are the same, that is, the conjectures have converged. The Schelling point is Player 1 plays A and Player 2 plays E, which is also the Pareto dominant outcome in Table 7 and the natural focal point of the best response equivalent identical interest game.

3. A sequential incomplete information game

Figure 1 below gives the extensive form representation of a pure coordination game.

There are 2 pure strategy perfect Bayesian equilibriums, both separating.

1. Type 1 player 1 plays R and type 2 player 1 plays L. Upon observing L player 2 plays U and upon observing R player 2 plays U.
2. Type 1 player 1 plays L and type 2 player 1 plays R. Upon observing L player 2 plays D and upon observing R player 2 plays D.

There are no off-equilibrium beliefs. The second equilibrium Pareto dominates the first equilibrium and is the Schelling point.

To find the Schelling point by convergence of conjectures, let the probability that type 1 sender plays L be a and the probability that type 2 sender plays L be b. Let the probability that the receiver plays U when L is observed be x and the probability that the receiver plays D when R is observed be y.

Fig. 1.

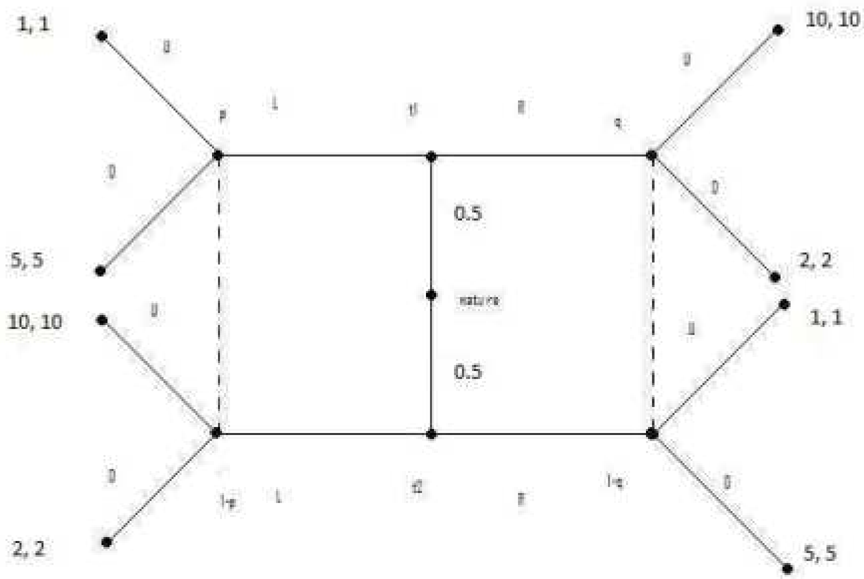


Table 13.

Order ofConjectures	a	b	x	y
1	1/2	1/2	1/2	1/2
2	0	1	1	1
3	0	1	1	1

The process of the convergence of conjectures is presented below: Figure 2 below gives the extensive form of a pure conflict game and there is no pure strategy Perfect Bayesian equilibrium nor Schelling point.

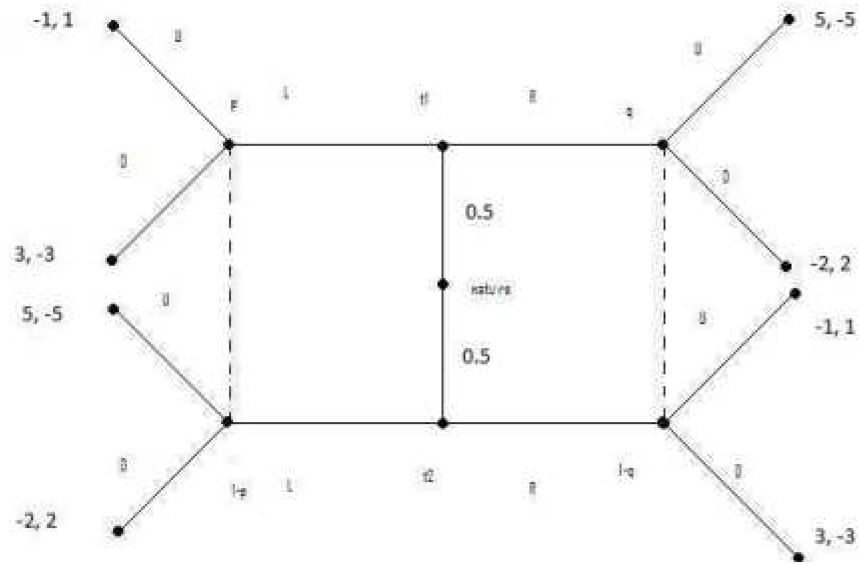
To find the Schelling point, let the probability that type 1 sender plays L be a and the probability that type 2 sender plays L be b. Let the probability that the receiver plays U when L is observed be x and the probability that the receiver plays D when R is observed be y.

The process of the updating of conjectures is presented below: Please note that

Table 14.

Order of Conjectures	a	b	x	y
1	1/2	1/2	1/2	1/2
2	0	1	0	0
3	1	0	0	0
4	1	0	1	1
5	0	1	1	1
6	0	1	0	0

Fig. 2.



the 6^{th} order conjectures are the same as those of the 2^{nd} order. Therefore the process does not converge and there is no Schelling point.

Schelling point is useful in the solution of incomplete information sequential games with both elements of conflicts and coordination. An example is provided below, the famous beer and quiche game.[5] Figure 3 below gives the extensive form of the game.

There are two perfect Bayesian equilibriums:

1. Both wimpy type sender and surly type sender play quiche. Upon observing quiche, the receiver plays Not and, upon observing beer, the receiver plays Duel. The off equilibrium belief is $q>0.5$.
2. Both wimpy type sender and surly type sender play beer. Upon observing quiche, the receiver plays Duel and, upon observing beer, the receiver plays Not. The off equilibrium belief is $p>0.5$.

The second equilibrium is ruled out by the intuitive criterion.

The Schelling point solution is presented below:

Let the probability that wimpy type sender chooses quiche be x and the probability that surly type sender chooses quiche be y. Let the probability that receiver plays duel when quiche is observed be u and the probability that the receiver plays duel when beer is observed be v.

The Schelling point is equilibrium 1, the equilibrium selected by the intuitive criterion.

4. A two-sided incomplete information game

This section shows that Schelling point is a good way of solving more complicated games, such as games with multiple sided incomplete information. Consider the two

[5] Refer to Gibbon (1992) p. 238, figure 4.4.3.

Fig. 3.

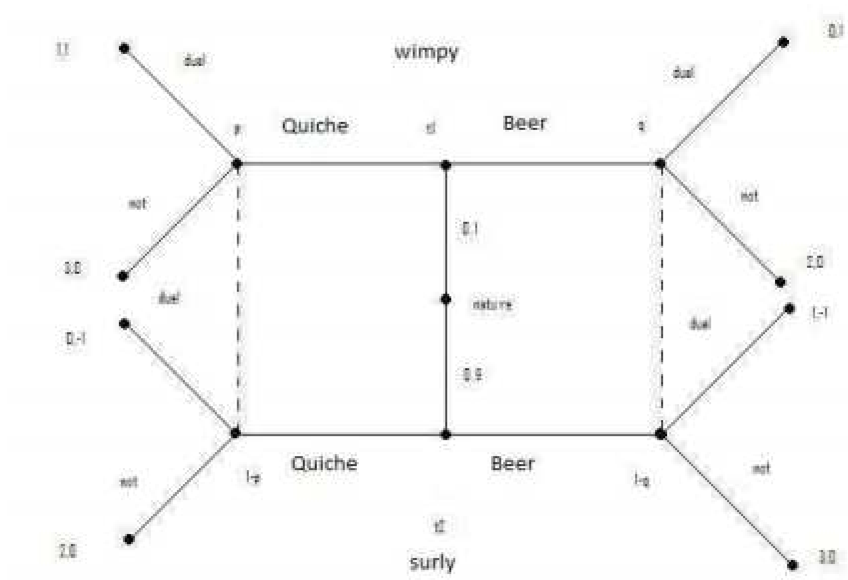


Table 15.

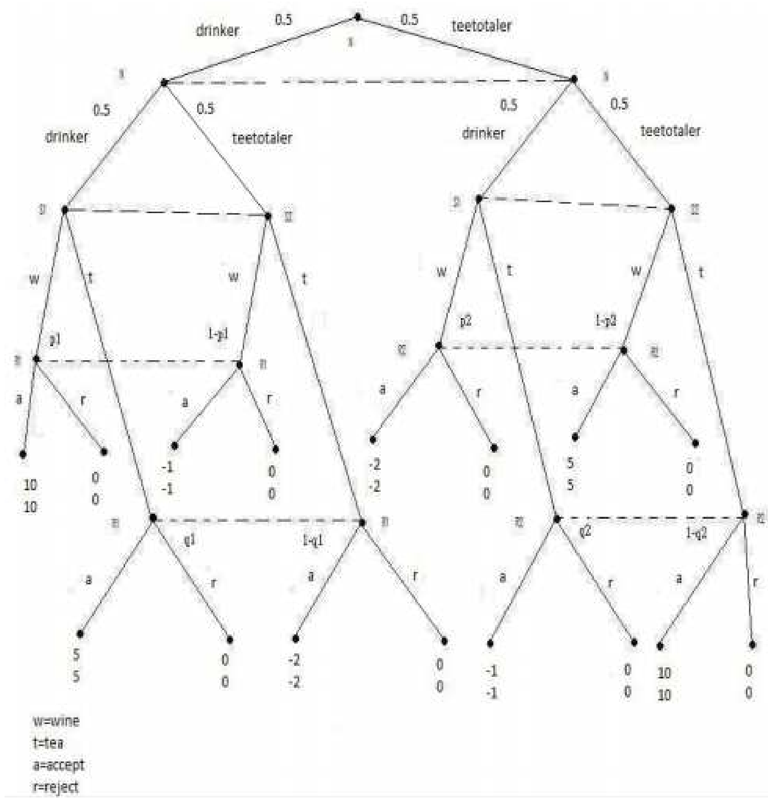
Order of Conjectures	x	y	u	v
1	1/2	1/2	1/2	1/2
2	1	0	0	0
3	1	0	1	0
4	0	0	1	0
5	0	0	1	0

sided incomplete information game represented extensively in Figure 3. It is a pure coordination game since the payoffs of player 1 and player 2 are identical. Solving by the concept of Perfect Bayesian Equilibrium is cumbersome and gives multiple equilibriums. In contrast, solving by Schelling point is swift and results in a unique equilibrium.

The four perfect Bayesian equilibriums of the above game are:

1. Drinker type player 2 plays wine. Teetotaler type player 2 plays wine. Upon observing wine, drinker type player 1 plays accept. Upon observing tea, drinker type player 1 plays reject. Upon observing wine, teetotaler type player 1 plays accept. Upon observing tea, teetotaler type player plays reject.
2. Drinker type player 2 plays tea. Teetotaler type player 2 plays tea. Upon observing wine, drinker type player 1 plays reject. Upon observing tea, drinker type player 1 plays accept. Upon observing wine, teetotaler type player 1 plays reject. Upon observing tea, teetotaler type player plays accept.
3. Drinker type player 2 plays wine. Teetotaler type player 2 plays tea. Upon observing wine, drinker type player 1 plays accept. Upon observing tea, drinker type player 1 plays reject. Upon observing wine, teetotaler type player 1 plays reject. Upon observing tea, teetotaler type player plays accept.

Fig. 4.



4. Drinker type player 2 plays tea. Teetotaler type player 2 plays wine. Upon observing wine, drinker type player 1 plays reject. Upon observing tea, drinker type player 1 plays accept. Upon observing wine, teetotaler type player 1 plays accept. Upon observing tea, teetotaler type player plays reject.

The Schelling point of the above game is PBE 3 which Pareto dominates the other three PBEs. To find the Schelling point of the game, let the probability that drinker type player 2 plays wine be p. Let the probability that teetotaler type player 2 plays wine be q. Let the probability that upon observing wine drinker type player 1 plays accept be u. Let the probability that upon observing tea, drinker type player 1 plays accept be v. Let the probability that upon observing wine teetotaler type player 1 plays accept be x. Let the probability that upon observing tea teetotaler type player plays accept be y.

The solution by convergence of conjectures to arrive at the Schelling point is presented below:

Table 16.

Order of Conjectures	p	q	u	v	X	y
1	1/2	1/2	1/2	1/2	1/2	1/2
2	1	0	1	1	1	1
3	1	0	1	0	0	1
4	1	0	1	0	0	1

5. Conclusions

As pointed out by Schelling (1960), in many situations of strategic interactions, there are both elements of conflict of interests and common interests. The key to the solution of such situations is to coordinate actions so that common interests could be maximized despite conflict of interests. The convergence of conjectures approached introduced here offers a way to find the Schelling point that could maximized common interests despite existence of conflict of interests. The solution concept would also eliminate unstable Nash equilibriums as these could not be the point where conjectures could converge. The solution concept also has the advantage that it could solve a larger set of games than the current approach based on Nash equilibrium, such as multiple sided incomplete information games.

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