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A Dynamic Oligopoly Marketing Model of Advertising

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Abstract We consider a dynamic oligopoly advertising model for both noncooperative and cooperative setting. Feedback Nash equilibrium strategies and cooperative strategies are found to determine the optimal advertising efforts of each firm for both setting respectively. Besides, depending upon the cooperative strategies, imputation is introduced as an optimal allocation of joint payoff and Imputation Distribution Procedure is used to guarantee the time consistency for cooperation.

Keywords: Advertising competition, Optimal control, Dynamic programming, Time consistency.

1. Introduction

267 billions of dollars were spent in advertising by the world's 100 largest advertisers in 2016 according to the statistics from the source: Ad Age Datacenter, it's easy to imagine that a company couldn't survive in the market without advertising. Hence, advertising as a strategy for market share competition has been studied in many marketing models, so as we are doing in this paper, where we are concentrating on the dynamic advertising model. Here each firm's market share depends on its own and its competitors' current and past advertising expenditures. Competition between firms is formulated by the noncooperative differential game as it was done by (Erickson, 2003), (Sorger, 1989), (JΓёrgensen and Zaccour, 2004), (Prasad and Sethi, 2004), (Naik et al., 2008) and (Prasad et al., 2009). In this paper we derived a dynamic oligopoly advertising model from (Prasad et al., 2009), in the paper the Sethi model was extended to model awareness of auto brands with churn term, which is the extension of the decay of market share term in monopoly models capturing forgetting and noise. Closed-loop Nash equilibrium concept is used to obtain the optimal advertising expenditure for noncooperative game. Similar to this paper's work, we also consider the noncooperative dynamic game and use feedback Nash equilibrium concept to obtain the optimal advertising expenditure, but more than that, the cooperative dynamic game which wasn't considered by the previous work was elicited to this model and mainly studied in our paper.

Noncooperative game theory deals with strategic interactions among multiple decision makers with the objective functions depending on the choices of all the players and suggests solution concepts for a case when players do not cooperate or make any arrangements about their actions. A player cannot simply optimize her own objective function independent from the choices of the other players. In 1950 and 1951 in (Nash, 1950),(Nash, 1951) by John Nash, such a solution concept was introduced, which is now called the Nash equilibrium. For differential game models Nash equilibrium can be defined in an open-loop strategies or in closed-loop strategies. For both types consult (Basar and Olsder, 1999). In this paper we use the approach with feedback strategies as the most preferable for the game theoretical problems.

The cooperative dynamic game theory offers socially convenient and group efficient solutions to different decision problems involving strategic actions. One of the fundamental questions in the theory of cooperative dynamic games is the formulation of optimal behavior for players. A characteristic function of a coalition is an essential concept in the theory of cooperative games. This function is defined as indicated in (Chander and Tulkens, 1995) as total payoff of players from coalition S in Nash equilibrium in the game with following set of players: coalition S (acting as one player) and players from the set $I \setminus S$. A computation of Nash equilibrium fully described in (Basar and Olsder, 1995) is necessary for this approach. A set of imputations or a solution of the game is determined by the characteristic function as the set of individually rational vectors. To guarantee the cooperation of all players will holds during the game, which means the cooperation is time consistent. Notion of time consistency was formalized mathematically by Petrosyan in the paper (Petrosjan, 1977). In the next paper on time consistency (Petrosyan and Danilov, 1979) L. Petrosyan defined the notion of imputation distribution procedure (IDP), which is used to compose a time consistent cooperative solution or single imputation. Later on, L. Petrosyan defined the notion of strong time consistency in the paper (Petrosyan, 1993), it was introduced to guarantee the time-consistency for a set-value cooperative solutions, such as Core, Nucleus etc.. See recent publications on this topic in (Petrosyan and Yeung, 2006), (Jorgensen and Yeung, 1999) and (Jorgensen et al., 2003). The property of time consistency introduced by Petrosyan in (Petrosyan, 1993) is examined for Shapley value.

The paper is organized as follows. In the next section, we formulate a dynamic oligopoly marketing model corresponding to the model from (Prasad et al., 2009). In section 3, feedback Nash equilibrium strategies for noncooperative setting are presented. In section 4, cooperative case is considered, Shapley value is used as cooperative solution corresponding to the optimal cooperative strategies and IDP is introduced to guarantee the time consistency property. In section 5, a numerical example is used to illustrate the results.

2. Initial model

Consider a n-firm oligopoly market in a mature product category so that the total sales of the category are relatively stable (Prasad et al., 2009). The advertising efforts as strategies are used by firms to compete on the oligopoly market, each firm tries to increase its market share while the competitors try to minimize it using the advertisement efforts. Denote by $x_i(t)$ the market share of firm $i \in I \equiv \{1, ..., n\}$ at time t and $n \geq 1$.

Market share dynamics of firm i has the following form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{n}{n-1}\rho_i u_i \sqrt{1-x_i} - \frac{1}{n-1}\sum_{j \in I} \rho_j u_j \sqrt{1-x_j} - \delta(x_i - \frac{1}{n}). \qquad (1)$$

with $x_i(0) = z_i(0)$, where $z_i(0)$ is a positive constant.

Payoff function of firm $i \in I$ is defined by its profit:

$$K(x_{i0}, t_0, u_1, .., u_n) = \int_0^\infty e^{-r_i \tau}[m_i x_i(\tau) - (u_i(\tau))^2] d\tau,$$

where r_i is the discount rate of firm i. We transformed this differential game model to the corresponding discrete time game with finite horizon T using Finite Difference Method (FDM). Introduce the equally distributed grid points $(t_j)_{j=\overline{0,N}}$ given by $t_j = jh$ where N is an integer and the spacing h is given by $h = T/N$ and $x_k \simeq x(t_k)$ for all $k \in \{0, ..., N\}$. As a result, the market share dynamics of firm i for dynamic game model became

$$x_i^{k+1} = h(\frac{n}{n-1}\rho_i u_i^k \sqrt{1-x_i^k} - \frac{1}{n-1}\sum_{j\in I}\rho_j u_j^k \sqrt{1-x_j^k}) - (h\delta - 1)x_i^k + h\delta\frac{1}{n}, \quad (2)$$

with $x_i^0 = z_i^0$, where z_i^0 is a positive constant.

Payoff function of firm i in dynamic game model became:

$$K_i^0(x^0, u^0) = \sum_{l=0}^{N} m_i x_i^l - (u_i^l)^2. \quad (3)$$

Full list of notations is presented in Table 1.

Table 1. List of variables and parameters

Notation	Explanation
$x_i^k \in [0,1]$	Market share of firm $i \in I \equiv \{1, ..., n\}$ at stage k.
$u_i^k \geq 0$	Advertising effort rate by firm i at stage k.
$\rho_i > 0$	Advertising effectiveness parameter of firm i.
$\delta > 0$	Churn parameter.
$m_i > 0$	Industry sales multiplied by the per unit profit margin of firm i.
$C(u_i(t))$	Cost of advertising of firm i, parameterized as $(u_i(t))^2$.

There is also a logical consistency requirement that the sum of market shares should be equal to one on each stage, i.e.

$$\sum_{i\in I} x_i^k = 1, \ \ for \ k \in \{0, ..., N\},$$

where this requirement can be checked by summing up right-side of all firms' motion equations in (2).

3. Noncooperative game model

Consider a noncooperative case, where each firm acts individually. According to (Yeung and Petrosyan, 2012) one can determine the noncooperative strategies as a feedback Nash equilibrium for the game defined by (2) and (4).

Definition 1. A feedback Nash equilibrium (Yeung, 1994) is an n-tuple of feedback strategies $\{\bar{u}_1^k, \bar{u}_2^k, ..., \bar{u}_n^k\}$, for $k \in \{0, ..., N\}$, if for every possible initial condition x^k of player i the following inequality holds

$$K_i^k(x^k, \bar{u}_1^k, ..., \bar{u}_{i-1}^k, u_i^k, \bar{u}_{i+1}^k, ..., \bar{u}_n^k) \leq K_i^k(x^k, \bar{u}_1^k, ..., \bar{u}_{i-1}^k, \bar{u}_i^k, \bar{u}_{i+1}^k, ..., \bar{u}_n^k), \forall i \in I.$$

If there exists a feedback Nash equilibrium solution with the set of strategies $\{\bar{u}_i^k,\ for\ k \in \{0,...,N\}, i \in \{1,...,n\}\}$ for (2) and (4), denote an equilibrium payoff function for firm $i \in I$ over stage k to N by

$$V_i^k(x) = \sum_{l=k}^{N} m_i x_i^l - (\bar{u}_i^l)^2,$$
$$where\ x_i^k = x.$$

A frequently used way to characterize and derive a feedback Nash equilibrium of the game is a dynamic programming.

Theorem 1. *A set of strategies $\{\bar{u}_i^k,\ for\ k \in \{0,...,N\},\ i \in \{1,...,n\}\}$ provides a feedback Nash equilibrium to the game defined by* (2) *and* (4) *if there exist functions $V_i^k(x)$, for $i \in \{1,...,n\}$ and $k \in \{0,...,N\}$, such that the following recursive relations are satisfied:*

$$\begin{aligned} V_i^k(x) &= \max_{u_i^k \geq 0} \{m_i x_i^k - (u_i^k)^2 + V_i^{k+1}(x_i^k, \bar{u}_1^k, ..., \bar{u}_{i-1}^k, u_i^k, \bar{u}_{i+1}^k, ..., \bar{u}_n^k)\} \\ &= m_i x_i^k - (\bar{u}_i^k)^2 + V_i^{k+1}(x_i^k, \bar{u}_1^k, ..., \bar{u}_{i-1}^k, \bar{u}_i^k, \bar{u}_{i+1}^k, ..., \bar{u}_n^k), \end{aligned} \tag{4}$$

$$V_i^{N+1}(x) = 0.$$

This theorem can be proved in the same way as it was done in (Yeung and Petrosyan, 2012).

Proposition 1. *The game equilibrium value functions in* (5) *are*

$$V_i^k(x_i^k) = \sum_{i \in I} A_i^k x_i^k + B^k(i), i \in \{1, ..., n\},\ k \in \{0, ..., N\},$$

where A_i^k, $B^k(i)$ are determined from the relations:

$$\begin{aligned} A_i^k &= m_i - (G_i^{k+1} Z_i)^2 - A_i^{k+1}(\delta h - 1), \\ A_j^k &= -2(G_j^{k+1} Z_j)^2 - A_j^{k+1}(\delta h - 1),\ j \neq i,\ j \in I \setminus i, \\ B^k(i) &= \sum_{i \in I} [2(G_i^{k+1} Z_i)^2 + \frac{A_i^{k+1} h\delta}{n}] + B^{k+1}(i) - (G_i^{k+1} Z_i)^2, \end{aligned}$$

with initial conditions $A_i^{N+1} = B^{N+1}(i) = 0$, $i \in \{1, ..., n\}$ and where $G_i^{k+1} = nA_i^{k+1} - \sum_{j \in I} A_j^{k+1}$, $Z_i = \frac{h\rho_i}{2(n-1)}$.

Corresponding feedback Nash equilibrium strategies are

$$\bar{u}_i^k = G_i^{k+1} Z_i \sqrt{1 - x_i^k}, k \in \{0, ..., N\}, i \in \{1, ..., n\}. \tag{5}$$

Proof see Appendix.

Substituting (6) into the original dynamics (2), we derive trajectory corresponding to the feedback Nash equilibrium, $i \in \{1, ..., n\}$ and $k \in \{0, ..., N\}$,

$$\bar{x}_i^{k+1} = nP_i^{k+1}(1 - \bar{x}_i^k) - \sum_{j \in I} [P_j^{k+1}(1 - \bar{x}_j^k)] + \frac{1}{n}, \tag{6}$$

where $P_i^{k+1} \equiv 2G_i^{k+1}(Z_i)^2 + \frac{h\delta - 1}{n}$. The noncooperative trajectory vectors for any stage are denoted by

$$\bar{x}^k = [\bar{x}_1^k, ..., \bar{x}_n^k], k = \overline{0, N}.$$

4. Cooperative game

Consider the case when all firms agree to cooperate. Maximizing the playersвЂ™ joint payoff guarantees grand coalition optimality in a game where payoffs are transferable. To maximize their joint payoff the following optimization should be performed:

$$\sum_{i\in I} K_i^0(x^0,u^0) = \sum_{i\in I}\sum_{l=0}^{N} m_i x_i^l - (u_i^l)^2 \to \max_{u_1,\dots,u_n} \tag{7}$$

subject to

$$x_i^{k+1} = h(\frac{n}{n-1}\rho_i u_i^k\sqrt{1-x_i^k} - \frac{1}{n-1}\sum_{j\in I}\rho_j u_j^k\sqrt{1-x_j^k}) - (h\delta-1)x_i^k + h\delta\frac{1}{n},\ x_i^0 = z_i^0.$$

This is a discrete time optimization problem. Dynamic programming (Bellman, 1957) is used to solve it. Suppose that the Bellman function has the following form:

$$W^k(x_k) = \max_{u_1^k,\dots,u_n^k}\left\{\sum_{i\in I}\sum_{l=k}^{N} m_i x_i^l - (u_i^l)^2\right\}, k\in\{0,\dots,N\}. \tag{8}$$

Theorem 2. *Assume that there exists function $W^k(x)$, such that the following recursive relations are satisfied:*

$$W^k(x_k) = \max_{u_1^k,\dots,u_n^k}\left\{\sum_{i\in I} m_i x_i^k - (u_i^k)^2 + W^{k+1}(x_k)\right\}, k\in\{0,\dots,N\}, \tag{9}$$

where $W^{N+1}(x_{N+1}) = 0$.

We use the following linear functions, observing that they satisfy the Bellman equations, to solve this recursive relations.

Proposition 2. *Bellman function $W^k(x_k)$ in (9) can be computed in the form*

$$W^k(x_k) = \sum_{i\in I} C_i^k x_i^k + D_i^k, \tag{10}$$

where C_i^k, D_i^k, $i\in\{1,\dots,n\}$ and $k\in\{0,\dots,N\}$ satisfy:

$$C_i^k = m_i - (Q_i^{k+1}Z_i)^2 - C_i^{k+1}(\delta h-1),$$
$$D_i^k = (Q_i^{k+1}Z_i)^2 + \frac{C_i^{k+1}\delta h}{n} + D_i^{k+1},$$

with initial conditions $C_i^{N+1} = D_i^{N+1} = 0$, $i\in\{1,\dots,n\}$.

The optimal cooperative strategies can be obtained as follows:

$$\varphi_i^k = Q_i^{k+1} Z_i\sqrt{1-x_i^k}, \forall i\in I,$$

where $Q_i^{k+1} = nC_i^{k+1} - \sum_{j\in I} C_j^{k+1}$.

Substituting optimal cooperative strategies in (2), cooperative trajectory of $i \in I$ is computed as

$$x_i^{c,k+1} = 2nQ_i^{k+1}Z_i^2(1-x_i^{c,k}) - \sum_{j\in I} 2Q_j^{k+1}Z_j^2(1-x_j^{c,k}) - (h\delta-1)x_i^{c,k} + \frac{h\delta}{n},$$

where $k \in \{0,\dots,N\}$.

It's worth mentioning that the advertising effort rate of each firm is equal to zero when all the firms are symmetric, because of the whole market acts as a monopoly when all symmetric firms decide to cooperate. Once consumers want to buy goods they have to buy it from one of the firm even if firms do not spend on advertising.

Proposition 3. *If all firms are identical: $m_i = m$, $\rho_i = \rho$ for $i \in I$, then the optimal strategies of players are*

$$\varphi_i^k = 0, \forall i \in I,\ k \in \{0,\dots,N\}.$$

Proof see Appendix.

4.1. Characteristic function

For each coalition $S \subset I$ define the values of characteristic function as it was done in (Chander and Tulkens, 1995):

$$V(S;x^c) = \begin{cases} \sum_{i\in I} K_i(x^c;u), & S = I, \\ \tilde{V}(S,x^c), & S \subset I, \\ 0, & S = \emptyset, \end{cases} \tag{11}$$

where $\tilde{V}(S,x^c)$ is defined as total payoff of players from coalition S in feedback Nash equilibrium $\bar{u} = (\bar{u}_1,...,\bar{u}_n)$ in the game with the following set of players: coalition S (acting as one player) and players from the set $|I \setminus S|$, i.e. in the game of $|I \setminus S| + 1$ players.

For $|S| = 1$ cooperative case becomes to the noncooperative case:

$$V^k(\{i\};x_k^c) = \alpha * V_i^k(x_i^{c,k}) = A_i^k x_i^{c,k} + B^k(i)\ ,\ i \in I \setminus S,$$

we consider a parameter α in this case which is a given discount of characteristic function of $|S| = 1$ under the cooperation . Parameter α guaranties the property of essential always hold for cooperative case, therefore α should satisfy this inequality:

$$0 < \alpha \le \frac{V^k(I;x_k^c)}{\sum_{i\in I} V_i^k(x_i^{c,k})}$$

In accordance with the formula (11) characteristic function for coalition $S = I$ is defined as

$$V^k(I;x_k^c) = W^k(x_k^c).$$

The characteristic function for coalition $S \subset I$ is defined as the payoff of players from coalition S in Nash feedback equilibrium in the game with $|I \setminus S + 1|$ players.

Corresponding system of Bellman equations for coalition S and players $i \in I \setminus S$ will have the form:

$$\begin{cases} V^k(S, x^k) = \max_{u^k \in S} \left\{ \sum_{i \in S} [m_i x_i^k - (u_i^k)^2] + V^{k+1}(S, x^{k+1}) \right\}, \\ V^k(\{i\}, x_i^k) = \max_{u_i^k} \left\{ m_i x_i^k - (u_i^k)^2 + V^{k+1}(\{i\}, x_i^{k+1}) \right\}, \ i \in I \setminus S, \end{cases} \tag{12}$$

with initial conditions $V^{N+1}(S, x^{N+1}) = 0, V^{N+1}(\{i\}, x_i^{N+1}) = 0, \ i \in I \setminus S$.

Proposition 4. *Bellman function for coalition S and players $i \in I \setminus S$ can be computed in the form:*

$$V^k(S, x^k) = \sum_{i \in I} \bar{C}_i^k x_i^k + E^k,$$

$$V^k(\{i\}, x_i^k) = \sum_{i \in I} \bar{A}_i^k x_i^k + \bar{B}(i)^k, \ i \in I \setminus S,$$

where parameters $\bar{C}_i^k$, $\bar{A}_i^k$,E^k and $\bar{B}(i)^k$ are defined as:

$$\begin{aligned}
\bar{C}_i^k &= m_i - (R_i^{k+1} Z_i)^2 - \bar{C}_i^{k+1}(\delta h - 1), \ i \in S, \\
\bar{C}_i^k &= -2\bar{G}_i^{k+1} R_i^{k+1} Z_i^2 - \bar{C}_i^{k+1}(\delta h - 1), \ i \in I \setminus S, \\
E^k &= \sum_{i \in S} (R_i^{k+1} Z_i)^2 + \sum_{i \in I \setminus S} 2\bar{G}_i^{k+1} R_i^{k+1} Z_i^2 + \sum_{i \in I} \frac{\bar{C}_i^{k+1} h \delta}{n} + E^{k+1}, \\
\bar{A}_i^k &= m_i - (\bar{G}_i^{k+1} Z_i)^2 - \bar{A}_i^{k+1}(\delta h - 1), i \in I \setminus S, \\
\bar{A}_j^k &= -2(\bar{G}_j^{k+1} Z_j)^2 - \bar{A}_j^{k+1}(\delta h - 1), \ j \in I \setminus (S \cup i), \\
\bar{A}_f^k &= -2\bar{G}_f^{k+1} R_f^{k+1} Z_f^2 - \bar{A}_f^{k+1}(\delta h - 1), \ f \in S, \\
\bar{B}(i)^k &= \sum_{i \in I \setminus S} 2(\bar{G}_i^{k+1} Z_i)^2 + \sum_{i \in S} 2\bar{G}_i^{k+1} R_i^{k+1} Z_i^2 + \sum_{i \in I} \frac{\bar{A}_i^{k+1} \delta h}{n} + \bar{B}(i)^{k+1} - \\
&\quad - (\bar{G}_i^{k+1} Z_i)^2, i \in I \setminus S,
\end{aligned}$$

with initial conditions $\bar{C}_i^{N+1} = 0$, $E^{N+1} = 0$, $\bar{A}_i^{N+1} = 0$ and $\bar{B}(i)^{N+1} = 0$, where

$$R_i^{k+1} = n\bar{C}_i^{k+1} - \sum_{j \in I} \bar{C}_j^{k+1},$$

$$\bar{G}_i^{k+1} = n\bar{A}_i^{k+1} - \sum_{j \in I} \bar{A}_j^{k+1}.$$

Optimal strategies used by players from coalition S and players $i \in I \setminus S$ can be computed as follows:

$$\bar{u}_i^{S,k} = (n\bar{C}_i^{k+1} - \sum_{j \in I} \bar{C}_j^{k+1}) \frac{h \rho_i \sqrt{1 - x_i^k}}{2(n-1)}, \ i \in S,$$

$$\bar{u}_i^k = (n\bar{A}_i^{k+1} - \sum_{j \in I} \bar{A}_j^{k+1}) \frac{h \rho_i \sqrt{1 - x_i^k}}{2(n-1)}, \ i \in I \setminus S.$$

Value of characteristic function for coalition $S \subset I$ is calculated as follows:

$$V^k(S; x^c) = V^k(S, x^{c,k}),$$

where $k \in \{0, \dots, N\}$.

4.2. Imputation set

Main problem in the theory of cooperative games is to construct realizable principle of allocation of total payoff $W^k(x_k^c)$ among players. An optimality principle can be introduced if all players agree to allocate the total cooperative payoff along the cooperative trajectory according to an imputation.

Definition 2. Imputation is a vector $\xi^k(x_k^c) = [\xi_1^k(x_1^{c,k}), ..., \xi_n^k(x_n^{c,k})]$ for $k = \overline{0, N}$, which satisfies the conditions

$$\xi_i^k(x_i^{c,k}) \geq V^k(\{i\}\,; x_k^c), i \in I,$$

$$\sum_{i \in I} \xi_i^k(x_i^{c,k}) = V^k(I; x_k^c).$$

4.3. Imputation distribution procedure

Following the continuous-time analysis of (Petrosyan and Yeung, 2006) for cooperative differential games, we formulate a discrete-time version of the Imputation Distribution Procedure so that the agreed upon imputations of definition 2 can be realized. By $B_i^k(x_i^{c,k})$ denote the payment that the firm i will receive at stage k under the cooperative agreement along the cooperative trajectory $\left\{x_i^{c,k}\right\}_k^N$.

Payment scheme involving $B_i^k(x_i^{c,k})$ constitutes an IDP in the sense that the imputation of firm i over the stages from k to N:

$$\xi_i^k(x_i^{c,k}) = \sum_{j=k}^{N} B_i^j(x_i^{c,j}). \tag{13}$$

4.4. Time consistency

The property of time-consistency was introduced in 1977 by Petrosyan (Petrosjan, 1977). To ensure stability in dynamic cooperation over time a stringent condition is required: the specific agreed-upon optimality principle must be maintained at any instant of time throughout the game along the optimal state trajectory. This condition is known as time consistency.

Definition 3. Solution $H(x_k^c, k)$ is called time consistent if for any imputation $\xi_i^k(x_i^{c,k}) \in H(x_k^c, k)$ exists IDP $B_i^k(x_i^{c,k})$, $k = \overline{0, N}$ such that it satisfies:

$$\sum_{k}^{N} B_i^k(x_i^{c,k}) \in H(x_k^c, k),$$

$$\sum_{k}^{N} B_i^k(x_i^{c,k}) = \xi_i^k(x_i^{c,k}).$$

Theorem 3. *If optimality principle satisfies the following conditions:*

(i) $u_i^k = \varphi_i^k$, *for* $i \in I$ *and* $k = \overline{0, N}$, *is the set of group optimal strategies for the this game,*

(ii) $B_i^k(x_i^k) = B_i^k(x_i^{c,k})$, *for* $i \in I$ *and* $k = \overline{0, N}$, *where*

$$B_i^k(x_i^{c,k}) = \xi_i^k(x^{c,k}) - \xi_i^{k+1}(x^{c,k+1}), \tag{14}$$

then $[\xi_1^k(x_1^{c,k}), ..., \xi_n^k(x_n^{c,k})]$ *is is time consistent.*

This theorem was proved in (Petrosjan and Yeung, 2012).

4.5. Cooperative solution

In this paper, we consider Shapley value as a cooperative solution $Sh^k(x^{c,k})$ (Shapley, 1953), which is calculated in the following way:

$$Sh_i^k(x^{c,k}) = \sum_{S \subset I, i \in S} \frac{(|I| - |S|)!(|S| - 1)!}{|I|!} \cdot [V^k(S; x^{c,k}) - V^k(S \setminus i; x^{c,k})], \tag{15}$$

for $k = \overline{0, N}$.

5. Numerical example

We consider a specific three-firms oligopoly case within the stage interval N=8. To illustrate our model, let the parameters for each firms to be $\rho = [0.4, 0.5, 0.3]$, $h = 0.4$, $\alpha = 0.7$, $\delta = 0.09$, $m = [0.6, 1, 1.2]$ and for the initial conditions $z^0 = [0.3, 0.5, 0.2]$.

Fig.1. shows feedback Nash equilibrium strategies for noncooperative game and Fig.2. shows how the market share of each firm changes corresponding to their advertising expenditure.

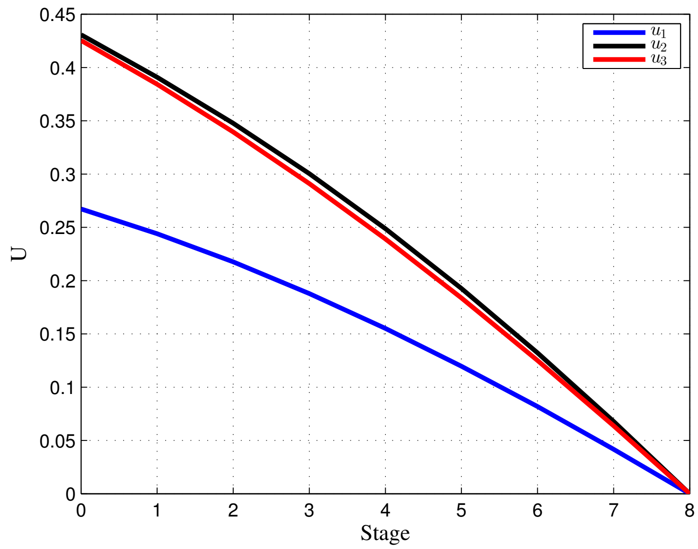

Fig. 1. Feedback Nash equilibrium strategies of noncooperative game

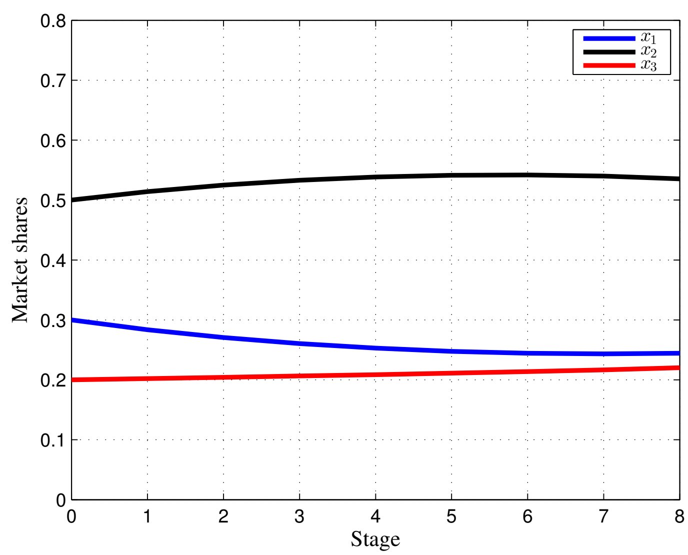

Fig. 2. Noncooperative trajectory

Noncooperative outcomes are obtained respectively, which is shown in Fig.3. The optimal cooperative strategies and cooperative trajectory are illustrated in Fig.4. and Fig.5.

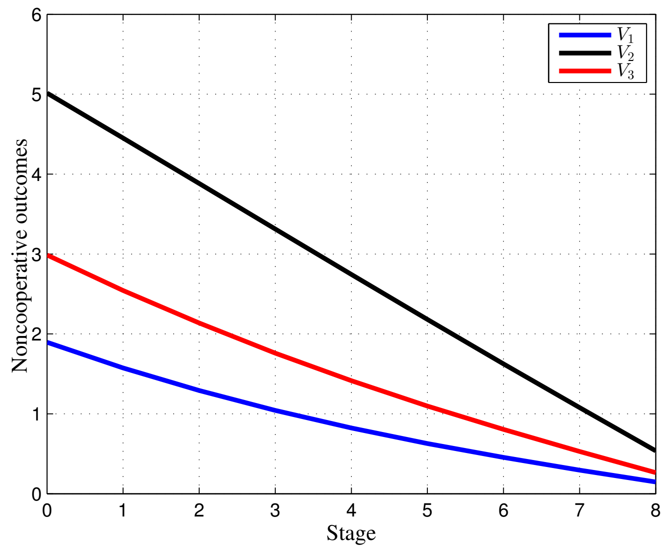

Fig. 3. Noncooperative outcomes

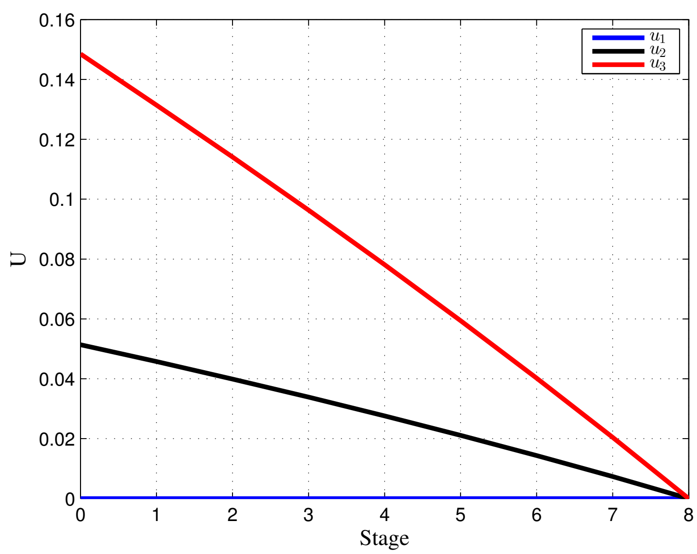

Fig. 4. The optimal strategies of cooperative game

On the Fig.6. Shapley value calculated using characteristic function (11) is presented.

In order to guarantee time consistency for cooperative game, Fig.7. shows IDP for Shapley value.

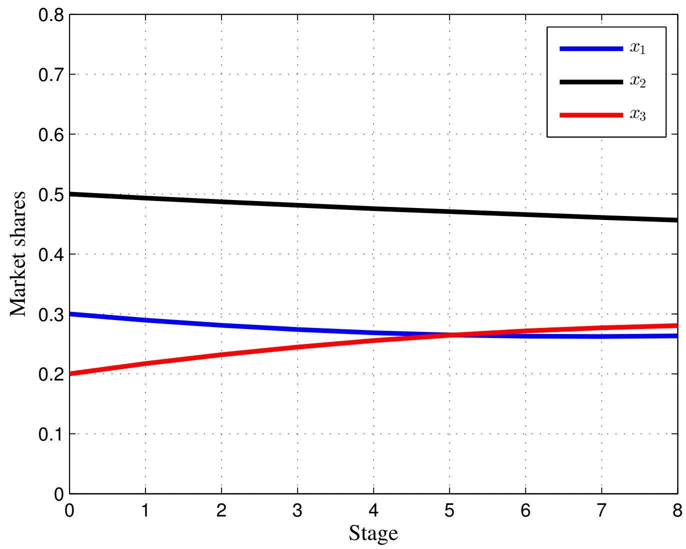

Fig. 5. Cooperative trajectory

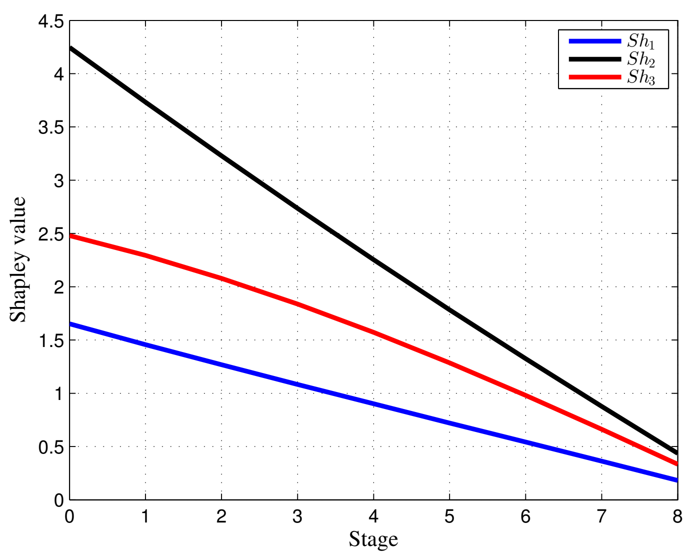

Fig. 6. Shapley value

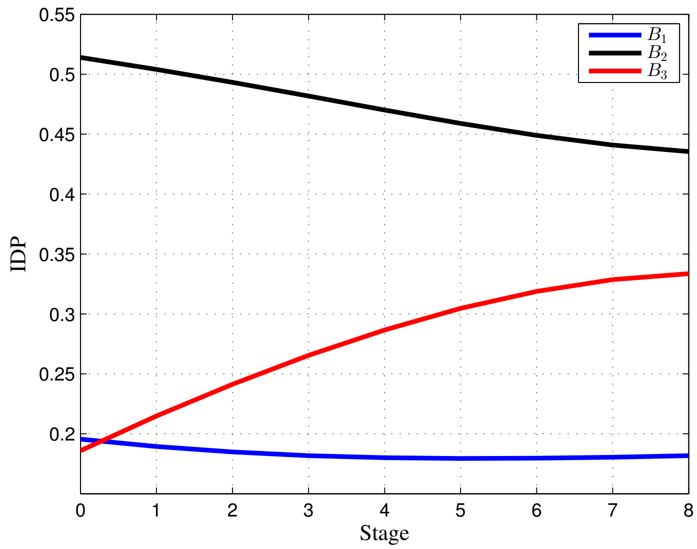

Fig. 7. Imputation Distribution Procedure

6. Conclusion

In this paper, we considered an oligopoly dynamic marketing model for the noncooperative game as in the previous paper, and obtained a feedback Nash equilibrium strategies for the advertising expenditures of all firms. Then the cooperation for all oligopolistic firms is introduced, the optimal cooperative strategies are obtained to determine the advertising expenditures, which maximize the profit for this cooperation, in order to maintain this cooperation for all firms. An agreement, namely Imputation Distribution Procedure, is obtained to guarantee no firms will collude and form a smaller coalition under this agreement.

A simple numerical example is presented in this paper to illustrate the results for both cooperative and noncooperative game obtained of the oligopoly dynamic marketing model.

Appendix

1. First appendix

The Bellman equation for firm i is given by

$$V_i^k(x_i^k) = \max_{u_i^k \geq 0} \left\{ m_i x_i^k - (u_i^k)^2 + V_i^{k+1}(x_i^{k+1}) \right\}.$$

Invoking (2) and $V_i^k(x_i^k) = \sum_{i\in I} A_i^k x_i^k + B^k(i)$, for $i \in I$ and $k \in \{0, ..., N\}$, then the above Bellman equation becomes

$$\begin{aligned}
&\sum_{i\in I} A_i^k x_i^k + B^k(i) = \max_{u_i^k \geq 0}\left\{ m_i x_i^k - (u_i^k)^2 + \sum_{i\in I} A_i^{k+1} x_i^{k+1} + B^{k+1}(i)\right\} \\
&= \max_{u_i^k \geq 0}\left\{ \sum_{i\in I} A_i^{k+1}[h(\frac{n}{n-1}\rho_i u_i^k \sqrt{1-x_i^k} - \frac{1}{n-1}\sum_{j\in I}\rho_j u_j^k \sqrt{1-x_j^k}) - \right. \\
&\left. -(h\delta - 1)x_i^k + h\delta\frac{1}{n}] + B^{k+1}(i) + m_i x_i^k - (u_i^k)^2 \right\}.
\end{aligned} \tag{16}$$

Performing the indicated maximization in above yields

$$-2u_i^k + \frac{h\rho_i}{n-1} n A_i^{k+1}\sqrt{1-x_i^k} - \frac{h\rho_i}{n-1}\sqrt{1-x_i^k}\sum_{j\in I} A_j^{k+1} = 0,$$

for $i \in I$ and $k \in \{0, ..., N\}$. Then the feedback Nash equilibrium of firm i can be obtained in the form

$$\bar{u}_i^k = max\{0, [nA_i^{k+1} - \sum_{j\in I} A_j^{k+1}]\frac{h\rho_i}{2(n-1)}\sqrt{1-x_i^k}\}.$$

Anticipating that the controls will be shown to be positive, substituting it into (16), collecting the terms together, then the parameters A_i^k and $B^k(i)$ can be expressed as

$$\begin{aligned}
A_i^k &= m_i - (G_i^{k+1} Z_i)^2 - A_i^{k+1}(\delta h - 1), \\
A_j^k &= -2(G_j^{k+1} Z_j)^2 - A_j^{k+1}(\delta h - 1),\ j \neq i,\ j \in I \setminus i, \\
B^k(i) &= \sum_{i\in I}[2(G_i^{k+1} Z_i)^2 + \frac{A_i^{k+1} h\delta}{n}] + B^{k+1}(i) - (G_i^{k+1} Z_i)^2,
\end{aligned}$$

where $G_i^{k+1} = nA_i^{k+1} - \sum_{j\in I} A_j^{k+1}$, $Z_i = \frac{h\rho_i}{2(n-1)}$.

2. Second appendix

Substituting (10) into (9) we receive

$$\begin{aligned}
W^k(x_k) &= \max_{u_1^k, ..., u_n^k}\left\{ \sum_{i\in I}[m_i x_i^k - (u_i^k)^2] + W^{k+1}(x_k)\right\} \\
\Rightarrow \sum_{i\in I} C_i^k x_i^k + D_i^k &= \max_{u_1^k, ..., u_n^k}\left\{ \sum_{i\in I}[m_i x_i^k - (u_i^k)^2] + \sum_{i\in I} C_i^{k+1} x_i^{k+1} + D_i^{k+1}\right\}
\end{aligned} \tag{17}$$

$$W^{N+1}(x_{N+1}) = 0.$$

substitute the right-side of (2) into (17). Performing the indicated maximization in (17) yields

$$-2u_i^k + (nC_i^{k+1} - \sum_{j\in I} C_j^{k+1})\frac{h\rho_i\sqrt{1-x_i^k}}{n-1} = 0.$$

The optimal cooperative strategies can be obtained in the form

$$\varphi_i^k = max\{0, (nC_i^{k+1} - \sum_{j\in I} C_j^{k+1})\frac{h\rho_i\sqrt{1-x_i^k}}{2(n-1)}\},$$

for $i \in I$ and $k \in \{0, ..., N\}$.

Anticipating that the cooperative strategies will be shown to be positive, then substituting them into (17), hence C_i^k and D_i^k are defined as:

$$C_i^k = m_i - (Q_i^{k+1} Z_i)^2 - C_i^{k+1}(\delta h - 1),$$
$$D_i^k = (Q_i^{k+1} Z_i)^2 + \frac{C_i^{k+1}\delta h}{n} + D_i^{k+1}.$$

3. Third appendix

The asymmetric optimal cooperative strategies can be obtained in the form

$$\varphi_i^k = max\{0, (nC_i^{k+1} - \sum_{j\in I} C_j^{k+1})\frac{h\rho\sqrt{1-x_i^k}}{2(n-1)}\},$$

for $i \in I$ and $k \in \{0, ..., N\}$.

Suppose that all firms are identical: $m_i = m$, $\rho_i = \rho$ for $i \in I$, then the related parameters in Proposition 2.3 can be rewritten as follows

$$C_i^k = m - (Q_i^{k+1} Z)^2 - C_i^{k+1}(\delta h - 1),$$
$$D_i^k = (Q_i^{k+1} Z)^2 + \frac{C_i^{k+1}\delta h}{n} + D_i^{k+1},$$

with initial conditions $C_i^{N+1} = D_i^{N+1} = 0$ and where $Z = \frac{h\rho}{2(n-1)}$ and $Q_i^{k+1} = nC_i^{k+1} - \sum_{j\in I} C_j^{k+1}$. It shows that the value of C_i^k is equal for $\forall i \in I$, $k \in \{0, ..., N\}$, which means the optimal cooperative strategies equal to 0 at any stage when all firms are identical.

4. Fourth appendix

Substituting the Bellman function of coalition S and individual players into (12) respectively and corresponding to (2). Performing the indicated maximization in (12) yields

$$-2u_i^k + (n\bar{C}_i^{k+1} - \sum_{j\in I} \bar{C}_j^{k+1})\frac{h\rho_i\sqrt{1-x_i^k}}{n-1} = 0,\ i \in S,$$
$$-2u_i^k + (n\bar{A}_i^{k+1} - \sum_{j\in I} \bar{A}_j^{k+1})\frac{h\rho_i\sqrt{1-x_i^k}}{n-1} = 0,\ i \in I \setminus S.$$

The optimal strategies can be obtained in the form

$$\bar{u}_i^{S,k} = max\{0, (n\bar{C}_i^{k+1} - \sum_{j\in I} \bar{C}_j^{k+1})\frac{h\rho_i\sqrt{1-x_i^k}}{2(n-1)}\},\ i \in S,$$

$$\bar{u}_i^{k} = max\{0, (n\bar{A}_i^{k+1} - \sum_{j\in I} \bar{A}_j^{k+1})\frac{h\rho_i\sqrt{1-x_i^k}}{2(n-1)}\},\ i \in I \setminus S.$$

Anticipating that the optimal strategies will be shown to be positive and substituting them into (12), then collecting the terms together we obtain the parameters $\bar{C}_i^k$, $\bar{A}_i^k$,E^k and $\bar{B}(i)^k$

$$\begin{aligned}
\bar{C}_i^k &= m_i - (R_i^{k+1}Z_i)^2 - \bar{C}_i^{k+1}(\delta h - 1),\ i \in S,\\
\bar{C}_i^k &= -2\bar{G}_i^{k+1}R_i^{k+1}Z_i^2 - \bar{C}_i^{k+1}(\delta h - 1),\ i \in I \setminus S,\\
E^k &= \sum_{i\in S}(R_i^{k+1}Z_i)^2 + \sum_{i\in I\setminus S} 2\bar{G}_i^{k+1}R_i^{k+1}Z_i^2 + \sum_{i\in I}\frac{\bar{C}_i^{k+1}h\delta}{n} + E^{k+1},\\
\bar{A}_i^k &= m_i - (\bar{G}_i^{k+1}Z_i)^2 - \bar{A}_i^{k+1}(\delta h - 1), i \in I \setminus S,\\
\bar{A}_j^k &= -2(\bar{G}_j^{k+1}Z_j)^2 - \bar{A}_j^{k+1}(\delta h - 1),\ j \in I \setminus (S \cup i),\\
\bar{A}_f^k &= -2\bar{G}_f^{k+1}R_f^{k+1}Z_f^2 - \bar{A}_f^{k+1}(\delta h - 1),\ f \in S,\\
\bar{B}(i)^k &= \sum_{i\in I\setminus S} 2(\bar{G}_i^{k+1}Z_i)^2 + \sum_{i\in S} 2\bar{G}_i^{k+1}R_i^{k+1}Z_i^2 + \sum_{i\in I}\frac{\bar{A}_i^{k+1}\delta h}{n} + \bar{B}(i)^{k+1} -\\
&- (\bar{G}_i^{k+1}Z_i)^2, i \in I \setminus S.
\end{aligned}$$

References

Bellman, R. (1957). *Dynamic Programming.* Princeton University Press: Princeton, NJ.

Basar, T. and Olsder, G. J. (1999). *Dynamic noncooperative game theory,* 2nd edition. Classics in Applied Mathematics, **23**, SIAM, Philadelphia.

Yeung, D. W. K. (1994). *On Differential Games with a Feedback Nash Equilibrium.* Journal of Optimization Theory and Applications, **82(1)**, 181–188.

Yeung, D. W. K. and L. A. Petrosyan (2012). *Subgame-consistent Economic Optimization.* New York, Springer.

Erickson, G. M. (2003). *Dynamic Models of Advertising Competition.* 2E, Kluwer, Norwell, MA.

Nash, J.F. Jr. (1950). *Equilibrium points in n-person games.* Proceedings of the National Academy of Sciences, **36(1)**, 48–49. doi:10.1073/pnas.36.1.48.

Nash, J.F. Jr. (1951). *Non-cooperative games.* Annals of Mathematics, **54(2)**, 286–295. doi:10.2307/1969529.

Petrosyan, L. A. and N. N. Danilov (1979). *Stability of solutions in non-zero sum differential games with transferable payoffs.* Vestnik Leningrad State University, **1**, 52–59.

Petrosyan, L. A. and D. W. K. Yeung (2006). *Dynamically stable solutions in randomly-furcating differential games.* Proceedings of the Steklov Institute of Mathematics, **253**, 208–220.

Petrosja, L. A. (1993). *Strongly time-consistent differential optimality principles.* Vestnik St. Petersburg University, Mathematics, **26**, 40–46.

Petrosyan, L. A. (1977). *Time-consistency of solutions in multi-player differential games.* Vestnik Leningrad State University, **4**, 46–52.

Petrosyan, L. A. and D. W. K. Yeung (2012). *Subgame Consistent Economic Optimization.* Birkhäuser Boston Incorporated.

Naik, P. A., A. Prasad and S. P. Sethi (2008). *Building brand awareness in dynamic oligopoly markets.* Management Science, **54(1)**, 129–138.

Chander, P. and H. Tulkens (1995). *A core-theoretic solution for the design of cooperative agree- ments on transfrontier pollution.* International Tax and Public Finance, **2(2)**, 279–293.

Prasad, A., Sethi, S. P., Naik, P. A. (2009). *Optimal control of an oligopoly model of advertising.* INCOM 2009, Moscow, Russia.

Prasad, A., Sethi, S. P. (2004). *Competitive advertising under uncertainty: A stochastic differential game approach.* Journal of Optimization Theory and Applications, **123(1)**, 163–185.

Sorger, G. (1989). *Competitive dynamic advertising: A modification of the Case game.* Journal of Economic Dynamics and Control, **13(1)**, 55–80.

Shapley, L. S. (1953). *A Value for N-Person Games, in Contributions to the Theory of Games II.* Kuhn, W. and Tucker, A.W., Eds., Princeton: Princeton Univ. Press, 307–317.

Jorgensen, S. and G. Zaccour (2004). *Differential Games in Marketing.* Kluwer, Academic Publishers, Boston, MA.

Jorgensen, S. and D. W. K. Yeung (1999). *Inter- and intragenerational renewable resource extraction.* Annals of Operations Research, **88**, 275–289.

Jorgensen, S., Martin-Herran, G. and G. Zaccour (2003). *Agreeability and Time Consistency in Linear-State Differential Games.* Journal of Optimization Theory and Applications, **119**, 49–63.

Basar, T. and G. J. Olsder (1995). *Dynamic Noncooperative Game Theory.* London, Academic Press.