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Acceptable Points in Games with Preference Relations

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Abstract For games with preference relations we introduce an acceptability concept. An outcome of a game is called *an acceptable one* if no players which have an objection to it in the form of some strategy (all of the required definitions are clarified in the introduction, see section 1). It is easy to show that every outcome at equilibrium point is an acceptable one but the converse is false. An aim of this article is a finding of conditions for existence of acceptable outcomes for games with preference relations (see sections 2 and 3). These conditions relate both to strategies and the preference relations of the players. The main requirements concerning the preference relations are acyclic and transitivity. It is a very important fact, that for game in which the sets of strategies of players are finite, the set of acceptable outcomes is non empty. For the class of games with payoff function acceptability condition is equivalent to individual rationality condition. An example of infinite game in which the set of acceptable outcomes is empty is given in section 4.

Keywords: game with preference relations, Nash equilibrium point, general equilibrium point, acceptable point.

1. Introduction

It is known that the equilibrium concept is the main game-theoretic optimality principle. However for realization of this principle we need in the introduction of mixed strategies. This fact is burdensome in terms of the applications of game theory what stimulates an investigation of other solution concepts.

In this article we study the so called acceptability concept for games with preference relations that is games in which a goal structure is given by binary relations on the set of possible outcomes. We consider acceptable situations and acceptable outcomes as optimal solutions in game with preference relations. Let us give precise definitions.

Formally, a game of n players with preference relations in the normal form can be given as a system of the type

$$G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F\rangle \tag{1}$$

where $N = \{1, \ldots, n\}$ is a set of *players*, $n \geq 2$; X_i is a set of *strategies* of the player i; A is a set of *outcomes*; $\rho_i \subseteq A^2$ is a *preference relation* for player i; F is a *realization function*, i.e. a mapping from the set of all *situations* $X = \prod_{i\in N} X_i$ into the set of outcomes A. We assume that $|X_i| \geq 2$ for all $i \in N$ and $|A| \geq 2$. A game G is called *finite one* if all sets X_i $(i \in N)$ are finite. In general case we suppose the binary relations ρ_i are reflexive and other properties must be indicated additionally. The correlation $a_1 \overset{\rho_i}{\lesssim} a_2$ means that the outcome a_2 is not less preferable than the outcome a_1 for player i. A game G is said to be a game with ordered (or quasi-ordered) outcomes if all $(\rho_i)_{i\in N}$ are order (respectively, quasi-order) relations.

Definition 1. A situation $x^0 = \left(x_i^0\right)_{i\in N}$ in the game G of the form (1) is called *Nash equilibrium point* if for all $i \in N$ and $x_i \in X_i$ the correlation

$$F\left(x^0 \parallel x_i\right) \overset{\rho_i}{\lesssim} F\left(x^0\right)$$

holds.

In the case when preference relations $(\rho_i)_{i\in N}$ not satisfy the linearity condition, we can consider a certain generalization of Nash equilibrium concept in the following manner.

Definition 2. A situation $x^0 = \left(x_i^0\right)_{i\in N}$ in game G is called *a general equilibrium point* if there does not exist $i \in N$ and $x_i \in X_i$ such that

$$F\left(x^0 \parallel x_i\right) \overset{\rho_i}{>} F\left(x^0\right).$$

Obviously, any Nash equilibrium point is a general equilibrium point also but the converse is false. In the case when all binary relations $(\rho_i)_{i\in N}$ satisfy the linearity condition these concepts are equivalent to each other.

We now consider a concept of *acceptable outcome* for game G of the form (1). Fix some $i \in N$ and put $X_{N\setminus i} = \prod\limits_{\substack{j\in N\\ j\neq i}} X_j$. It is evident that we can consider $X_{N\setminus i}$ as a set of strategies of the complementary coalition $N \setminus i$. A pair $\left(x_i, x_{N\setminus i}\right)$ where $x_i \in X_i$ and $x_{N\setminus i} \in X_{N\setminus i}$ uniquely defines some outcome in game G which is denoted by $F\left(x_i, x_{N\setminus i}\right)$.

Definition 3. We say that a strategy $x_i^0 \in X_i$ is an *objection of player i to outcome* $a \in A$ if for any strategy $x_{N\setminus i} \in X_{N\setminus i}$ of the complementary coalition the correlation $F\left(x_i^0, x_{N\setminus i}\right) \overset{\rho_i}{>} a$ holds. An outcome $a \in A$ is called *an acceptable one* for player i if he has not objections to it. An outcome a is called *acceptable one in game G* if this outcome is *acceptable* for all players $i \in N$.

Therefore an outcome $a \in A$ is an *acceptable* one in game G if for any $i \in N$ and $x_i \in X_i$ there exists a strategy $x_{N\setminus i} \in X_{N\setminus i}$ of the complementary coalition such that the condition $\neg\left(F\left(x_i, x_{N\setminus i}\right) \overset{\rho_i}{>} a\right)$ holds. Indicated strategy $x_{N\setminus i}$ of complementary coalition is called *a punishing strategy.*

Some strengthening of the acceptability concept is the following.

Definition 4. An outcome $a \in A$ is called *quite acceptable one* for player i if there exists a strategy $x_{N\setminus i} \in X_{N\setminus i}$ of complementary coalition such that for any $x_i \in X_i$ holds the condition $\neg\left(F\left(x_i, x_{N\setminus i}\right) \overset{\rho_i}{>} a\right)$. An outcome a is called *quite acceptable one in game G* if it is quite acceptable for all players $i \in N$.

These concepts are transferred from outcomes of game G to its situations. Namely, a situation $x \in X$ in game G is called *acceptable* (or *quite acceptable*) one if the outcome $F(x)$ is acceptable (or quite acceptable) respectively.

Remark 1. A general equilibrium point is a quite acceptable (and hence an acceptable) situation in game G with preference relations.

Indeed, let $x^0 = \left(x_i^0\right)_{i \in N}$ be a general equilibrium point in game G with preference relations. Put $x_{N\setminus i}^0$ be the projection of situation x^0 on $X_{N\setminus i}$. Using the definition 2, we obtain for any $i \in N$ and $x_i \in X_i$

$$\neg \left(F\left(x_i, x_{N\setminus i}^0\right) \overset{\rho_i}{>} F\left(x^0\right) \right).$$

Hence for each $i \in N$ the strategy $x_{N\setminus i}^0$ of the complementary coalition $N \setminus i$ is a punishing one and it does not depend on the deviation of player i. Therefore the outcome $F\left(x^0\right)$ is a quite acceptable one and the situation x^0 is a quite acceptable also.

Remark 2. Equilibrium points and acceptable situations are stable situations of game in the following sense. For acceptable situation, any player's deviation from its original strategy could be "punished" by the complementary coalition of other players. In the case of equilibrium point such punishment occurs when the omission of the other players, i.e. automatically. In the general case of acceptable situation the complementary coalition has only "circuit response" to every possible deviation of the player from his initial strategy (that is "stable based on threats" in terminology of H. Moulin, see Moulin, 1981). Finally, if a situation of a game is quite admissible, the choice of "punishment" by complementary coalition does not depend on the deviation of the player. Therefore in this case for complementary coalition it is sufficiently to know only the fact of deviation of a player from its original strategy.

Note that acceptable points in general cooperative n-person games with payoff functions was study by Aumann and Dreze, 1974. See also the monograph of Moulin, 1981.

2. Sufficient conditions for existence of acceptable outcomes

2.1. Games with acyclic preferences

Theorem 1. *Let G be a game with preference relations of the form (1) in which the sets of players strategies are finite. If for any $i \in N$ the preference relation ρ_i is acyclic then the set of acceptable outcomes in game G is not empty.*

Proof (of theorem 1). First suppose the set of outcomes in game G is finite. Denote by W_i the set of all outcomes to which player $i \in N$ has some objection, i.e.

$$W_i = \{a \in A\colon\ (\exists x_i \in X_i)\left(\forall x_{N\setminus i} \in X_{N\setminus i}\right) F\left(x_i, x_{N\setminus i}\right) \overset{\rho_i}{>} a\}. \tag{2}$$

The case 1: all $W_i \neq \varnothing$. Since according to our assumptions the set A is finite and preference relation ρ_i is acyclic then in graph of strict preferences $\langle A, \rho_i^* \rangle$ no infinite paths hence every non-empty subset of the set A has a maximal element (see Rozen, 2013). Fix for all $i \in N$ some maximal element a_i^* under preference relation ρ_i in the subset W_i. Because $a_i^* \in W_i$, we obtain using (2) that for every $i \in N$ there exists a strategy $x_i^0 \in X_i$ satisfying for any strategy $x_{N\setminus i} \in X_{N\setminus i}$ the correlation

$$F\left(x_i^0, x_{N\setminus i}\right) \overset{\rho_i}{>} a_i^*. \tag{3}$$

Consider the situation $x^0 = \left(x_i^0\right)_{i \in N}$. Since i-th component of this situation is the strategy x_i^0 then for situation x^0 the correlation (3) holds for all $i \in N$ i.e.

$$(\forall i \in N)\, F\left(x^0\right) \overset{\rho_i}{>} a_i^*. \tag{4}$$

Because element a_i^* is a maximal one in the subset W_i, it follows from (4) that $F\left(x^0\right) \notin W_i$ for all $i \in N$, i.e. the outcome $F\left(x^0\right)$ is an acceptable one for each player $i \in N$. Hence x^0 is an acceptable point in game G.

The case 2: $W_i \neq \varnothing$ for some $i \in N$. Put $N^0 = \{i \in N \colon W_i = \varnothing\}$ and $N^1 = \{i \in N \colon W_i \neq \varnothing\}$. Like in the case 1 we can fix some maximal element b_i^* under preference relation ρ_i in every non-void subset $W_i\left(i \in N^1\right)$. In accordance with (2) for every $i \in N^1$ there exists the strategy $x_i^1 \in X_i$ such that for any strategy $x_{N\setminus i}$ of complementary coalition $N \setminus i$ the condition $F\left(x_i^1, x_{N\setminus i}\right) \overset{\rho_i}{>} b_i^*$ holds. Now for every $i \in N^0$ fix arbitrary a strategy $x_i^1 \in X_i$. Then in situation $x^1 = \left(x_i^1\right)_{i\in N}$ for all $i \in N^1$ holds

$$F\left(x^1\right) \overset{\rho_i}{>} b_i^*. \tag{5}$$

Since element b_i^* is a maximal one under preference relation ρ_i in subset W_i, it follows from (5) the condition $F\left(x^1\right) \notin W_i$ i.e. the outcome $F\left(x^1\right)$ is an acceptable one for all players $i \in N^1$. Because for any $i \in N^0$ holds $W_i = \varnothing$ then every outcome in game G is acceptable for any player $i \in N^0$. Therefore the outcome $F\left(x^1\right)$ is acceptable for all players $i \in N$, i.e. $F\left(x^1\right)$ is an acceptable outcome and the situation x^1 is an acceptable one in game G.

It is shown the existence of acceptable situation (and acceptable outcome also) in assumption that the set of outcomes of game G is finite. Now consider the case when the set of outcomes in game G is infinite. Consider the game

$$G^0 = \left\langle N, (X_i)_{i\in N}, A^0, \left(\rho_i^0\right)_{i\in N}, F\right\rangle$$

where A^0 is the range of function F and for each $i \in N$ the preference relation ρ_i^0 is the restriction of relation ρ_i under subset A^0. Since in accordance with our assumption the sets of strategies of players are finite then subset A^0 is finite also and relations ρ_i^0 remains to be acyclic. As proved above, the game G^0 has some acceptable outcome $a^* \in A^0$. Let us show that the outcome a^* is an acceptable one in game G also. Indeed, in the opposite case there exists a player $i \in N$ and a strategy $x_i' \in X_i$ which is its objection to the outcome a^* in game G, i.e. holds

$$\left(\forall x_{N\setminus i} \in X_{N\setminus i}\right) F\left(x_i', x_{N\setminus i}\right) \overset{\rho_i}{>} a^*. \tag{6}$$

Since elements $F\left(x_i', x_{N\setminus i}\right)$ and a^* belong to the set A^0 then conditions

$$F\left(x_i', x_{N\setminus i}\right) \overset{\rho_i}{>} a^* \quad \text{and} \quad F\left(x_i', x_{N\setminus i}\right) \overset{\rho_i^0}{>} a^*$$

are equivalent. Then using (6) we obtain that the strategy $x_i' \in X_i$ is an objection of player i to outcome a^* in game G^0 which leads to contradiction. □

Corollary 1. *An antagonistic game with preference relations $G = \langle X, Y, A, F, \rho\rangle$ in which sets of strategies X, Y are finite and the preference relation ρ of player 1 is acyclic, has an acceptable situation (hence an acceptable outcome also). In particular, any finite antagonistic game with ordered outcomes has an acceptable outcome.*

For the proof it is sufficiently to remark that the acyclic condition for relation ρ implies the acyclic condition for inverse relation ρ^{-1}.

2.2. Games with quasi-ordered outcomes

In this section we consider n-person game $G = \langle N, (X_i)_{i \in N}, A, (\rho_i)_{i \in N}, F \rangle$ with quasi-ordered outcomes. Our aim is a finding of condition for existence of acceptable points in such game. For arbitrary $i \in N$ define β_i*-domination of strategies for player* $i \in N$ in game G by the equivalence

$$x_i^1 \overset{\beta_i}{\lesssim} x_i^2 \Leftrightarrow \left(F\left(x_i^2, X_{N \setminus i}\right)\right)^{\uparrow} \subseteq \left(F\left(x_i^1, X_{N \setminus i}\right)\right)^{\uparrow}. \tag{7}$$

Remark 3. We denote by $\left(F\left(x_i, X_{N \setminus i}\right)\right)^{\uparrow}$ the set of all majorant for subset

$$\left(F\left(x_i, X_{N \setminus i}\right)\right) \overset{df}{=} \{F\left(x_i, x_{N \setminus i}\right) : x_{N \setminus i} \in X_{N \setminus i}\}$$

under quasi-order ρ_i, i.e.

$$\left(F\left(x_i, X_{N \setminus i}\right)\right)^{\uparrow} = \{a \in A\colon \left(\exists x_{N \setminus i} \in X_{N \setminus i}\right) a \overset{\rho_i}{\gtrsim} F\left(x_i, x_{N \setminus i}\right)\}.$$

Note that subset $\left(F\left(x_i, X_{N \setminus i}\right)\right)^{\uparrow}$ is the dual ideal generated by x_i-row $F\left(x_i, X_{N \setminus i}\right)$ in quasi-ordered set $\langle A, \rho_i \rangle$.

Obviously, β_i-domination of strategies for player $i \in N$ is an quasi-ordering on X_i. The strict part and the symmetric part of quasi-order $\overset{\beta_i}{\lesssim}$ can be written respectively in the form

$$x_i^1 \overset{\beta_i}{<} x_i^2 \Leftrightarrow F\left(x_i^2, X_{N \setminus i}\right)^{\uparrow} \subset F\left(x_i^1, X_{N \setminus i}\right)^{\uparrow}; \tag{8}$$

$$x_i^1 \overset{\beta_i}{\sim} x_i^2 \Leftrightarrow F\left(x_i^1, X_{N \setminus i}\right)^{\uparrow} = F\left(x_i^2, X_{N \setminus i}\right)^{\uparrow}. \tag{9}$$

Theorem 2. *Let* $G = \langle N, (X_i)_{i \in N}, A, (\rho_i)_{i \in N}, F \rangle$ *be a game with quasi-ordered outcomes. Suppose that every player* $i \in N$ *uses its* β_i*-maximal strategy* $x_i^0 \in X_i$. *Then the situation* $x^0 = \left(x_i^0\right)_{i \in N}$ *is acceptable one and the outcome* $F\left(x^0\right)$ *also is an acceptable one in game* G.

A proof of theorem 2 is based on lemma 1 which has some independent interest.

Lemma 1. *Let* $x_i^0 \in X_i$ *be* β_i*-maximal strategy of player* i. *Then for any situation* $x \in X$ *the outcome* $F\left(x \parallel x_i^0\right)$ *is an acceptable one for player* i.

Proof (of lemma 1). Fix an arbitrary strategy $x_{N \setminus i}^0 \in X_{N \setminus i}$. We need to show that the outcome $F\left(x_i^0, x_{N \setminus i}^0\right)$ is an acceptable one for player i. Indeed, otherwise there exists a strategy $x_i^1 \in X_i$ such that for any $x_{N \setminus i} \in X_{N \setminus i}$ holds

$$F\left(x_i^1, x_{N \setminus i}\right) \overset{\rho_i}{>} F\left(x_i^0, x_{N \setminus i}^0\right). \tag{10}$$

Let us show the inclusion

$$\left(F\left(x_i^1, X_{N \setminus i}\right)\right)^{\uparrow} \subseteq \left(F\left(x_i^0, x_{N \setminus i}^0\right)\right)^{\uparrow}. \tag{11}$$

Indeed, assume $a \in \left(F\left(x_i^1, X_{N\setminus i}\right)\right)^{\uparrow}$ i.e. $a \overset{\rho_i}{\gtrsim} F\left(x_i^1, x_{N\setminus i}\right)$ for some $x_{N\setminus i} \in X_{N\setminus i}$. Using the transitivity of quasi-order ρ_i and (10) we obtain $a \overset{\rho_i}{\gtrsim} F\left(x_i^0, x_{N\setminus i}^0\right)$ then $a \in \left(F\left(x_i^0, x_{N\setminus i}^0\right)\right)^{\uparrow}$. Moreover since $\left(F\left(x_i^0, x_{N\setminus i}^0\right)\right)^{\uparrow} \subseteq \left(F\left(x_i^0, X_{N\setminus i}\right)\right)^{\uparrow}$ we have

$$\left(F\left(x_i^1, X_{N\setminus i}\right)\right)^{\uparrow} \subseteq \left(F\left(x_i^0, X_{N\setminus i}\right)\right)^{\uparrow}. \tag{12}$$

We now prove that in (12) the inverse inclusion is false. Indeed, otherwise because

$$F\left(x_i^0, x_{N\setminus i}^0\right) \in F\left(x_i^0, X_{N\setminus i}\right) \subseteq \left(F\left(x_i^0, X_{N\setminus i}\right)\right)^{\uparrow},$$

we obtain $F\left(x_i^0, x_{N\setminus i}^0\right) \in \left(F\left(x_i^1, X_{N\setminus i}\right)\right)^{\uparrow}$ i.e. $F\left(x_i^0, x_{N\setminus i}^0\right) \overset{\rho_i}{\gtrsim} F\left(x_i^1, x'_{N\setminus i}\right)$ for some $x'_{N\setminus i} \in X_{N\setminus i}$. On the other hand according with (10) we have the strict inequality $F\left(x_i^1, x'_{N\setminus i}\right) \overset{\rho_i}{>} F\left(x_i^0, x_{N\setminus i}^0\right)$ that contradicts the previous correlaton. Thus the strict inclusion $\left(F\left(x_i^1, X_{N\setminus i}\right)\right)^{\uparrow} \subset \left(F\left(x_i^0, X_{N\setminus i}\right)\right)^{\uparrow}$ holds and according with (8) we obtain $x_i^1 \overset{\beta_i}{>} x_i^0$ that contradicts the β_i-maximality condition of strategy x_i^0. Lemma 1 is proved. □

Theorem 2 is a direct consequence of lemma 1. Indeed, according with lemma 1 the situation $x^0 = \left(x_i^0\right)_{i\in N}$ in which each player $i \in N$ uses its β_i-maximal strategy x_i^0 is acceptable for all players N that is an acceptable situation in game G and the outcome $F\left(x^0\right)$ is an acceptable one also.

We now show some sufficient conditions for an existence of acceptable outcomes in game with quasi-ordered outcomes. These conditions are based on theorem 2.

Let $G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F\rangle$ be a game with quasi-ordered outcomes. Consider the following conditions concerning dual ideals in quasi-ordered set $\langle A, \rho_i\rangle$ $(i \in N)$.

(C1). *For each $i \in N$ there exists a strategy $x_i \in X_i$ such that any strict descending chain of dual ideals of the form*

$$\left(F\left(x_i, X_{N\setminus i}\right)\right)^{\uparrow} \supset \left(F\left(x_i^1, X_{N\setminus i}\right)\right)^{\uparrow} \supset \left(F\left(x_i^2, X_{N\setminus i}\right)\right)^{\uparrow} \supset \dots \tag{13}$$

is terminated at some finite number.

(C2). *For each $i \in N$ and strategy $x_i \in X_i$ any strict descending chain of dual ideals of the form (13) is terminated at some finite number.*

(C3). *For arbitrary $i \in N$ let $X_i^0 \subseteq X_i$ be some subset of strategies of player i such that for every $x'_i, x''_i \in X_i^0$ dual ideals $\left(F\left(x'_i, X_{N\setminus i}\right)\right)^{\uparrow}$ and $\left(F\left(x''_i, X_{N\setminus i}\right)\right)^{\uparrow}$ are comparable under inclusion. Then there exists a strategy $x_i^* \in X_i$ satisfying the condition*

$$\bigcap_{x_i\in X_i^0} \left(F\left(x_i, X_{N\setminus i}\right)\right)^{\uparrow} = \left(F\left(x_{i^*}, X_{N\setminus i}\right)\right)^{\uparrow}. \tag{14}$$

Theorem 3. *Assume for game $G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F\rangle$ with quasi-ordered outcomes at least one of conditions (C1)вЂ“(C3) holds. Then in game G there exists an acceptable situation and an acceptable outcome also.*

Proof (of theorem 3). Assume the condition (C1) holds. Suppose the chain (13) is terminated at some member $\left(F\left(x_i^m, X_{N\setminus i}\right)\right)^{\uparrow}$ where $x_i^m \in X_i$. Then $\left(F\left(x_i^m, X_{N\setminus i}\right)\right)^{\uparrow}$ is a maximal dual ideal of the form $\left(F\left(x_i', X_{N\setminus i}\right)\right)^{\uparrow}$, $x_i' \in X_i$ in quasi-ordered set $\langle A, \rho_i \rangle$. In accordance with (8) the strategy x_i^m is β_i-maximal strategy for player i and using theorem 2 we obtain the required statement. In the case when the condition (C2) holds, the proof is similar. Assume now that the condition (C3) satisfies. Then it follows from (11) that the quasi-ordered set $\left\langle X_i, \overset{\beta_i}{\lesssim} \right\rangle$ is inductive one and according with Zorn's lemma it has a maximal element. It remains to use theorem 2. □

3. Conditions for uniqueness of acceptable outcome

In this section we consider the uniqueness of acceptable outcome problem for games with quasi-ordered outcomes. Firstly note the following

Remark 4. Let $G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F\rangle$ be a game with quasi-ordered outcomes of the form (1). Consider the so-called *a natural equivalence relation* $\varepsilon = \bigcap\limits_{i\in N} \varepsilon_i$ where $\varepsilon_i = \rho_i \cap \rho_i^{-1}$. Since for every $i \in N$ the inclusion $\varepsilon \subseteq \rho_i$ holds, the conditions $a_1 \overset{\rho}{>} a_2$ and $a_2 \overset{\varepsilon}{\equiv} a_2'$ imply $a_1 \overset{\rho}{>} a_2'$ for all $a_1, a_2, a_2' \in A$. It follows that if some outcome $a \in A$ is acceptable one for player $i \in N$ then any outcome $a' \overset{\varepsilon}{\equiv} a$ is an acceptable one for player $i \in N$ also. Therefore the uniqueness of acceptable outcome in game G can be considered *"up to natural equivalence ε"* only.

Lemma 2. *Let x^0 be Nash equilibrium point in game G with quasi-ordered outcomes of the form (1). For any $i \in N$ define a set W_i consisting of strict guaranteed outcomes of player i:*

$$W_i = \{a \in A\colon\ (\exists x_i \in X_i)\left(\forall x_{N\setminus i} \in X_{N\setminus i}\right) F\left(x_i, x_{N\setminus i}\right) \overset{\rho_i}{>} a\}.$$

Then the following inclusion holds:

$$W_i \subseteq \{a \in A\colon a \overset{\rho_i}{<} F\left(x^0\right)\}. \tag{15}$$

Proof (of lemma 2). Assume $a \in W_i$ i.e. there exists a strategy $x_i^* \in X_i$ such that $F\left(x \parallel x_i^*\right) \overset{\rho_i}{>} a$ for any $x \in X$. Set $x = x^0$ and we get $F\left(x^0 \parallel x_i^*\right) \overset{\rho_i}{>} a$. On the other hand, since x^0 is Nash equilibrium point, the correlation $F\left(x^0 \parallel x_i^*\right) \overset{\rho_i}{\lesssim} F\left(x^0\right)$ holds. Because relation $\overset{\rho_i}{\lesssim}$ is transitive, it follows from last two correlations that $a \overset{\rho_i}{<} F\left(x^0\right)$ which was to be proved. □

Corollary 2. *Let x^0 be Nash equilibrium point in game G with quasi-ordered outcomes of the form (1). Then*

$$\bigcup_{j\in N} W_j \subseteq \bigcup_{i\in N} \{a \in A\colon a \overset{\rho_i}{<} F\left(x^0\right)\}. \tag{16}$$

Definition 5. Nash equilibrium point x^0 in game G is called *a special one* if in (16) the equality holds, i.e.

$$\bigcup_{j\in N} W_j = \bigcup_{i\in N} \{a \in A\colon a \overset{\rho_i}{<} F\left(x^0\right)\}. \tag{17}$$

Definition 6. Let A be an arbitrary set and a collection $(\rho_i)_{i\in N}$ of quasi-orders on A is given. An element $c \in A$ is called *a centric one* if for any $a \in A$ holds $a \overset{\varepsilon}{\equiv} c$ or $a \overset{\rho_i}{<} c$ for some $i \in N$, where ε is the natural equivalence relation.

It is easy to show the following statement.

Lemma 3. *Let $G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F\rangle$ be a game with quasi-ordered outcomes which has Nash equilibrium point x^0. Then $F\left(x^0\right)$ is an unique up to the natural equivalence ε acceptable outcome in game G if and only if the situation x^0 is a special one and element $F\left(x^0\right)$ is a centric.*

Lemma 3 gives a solution of the uniqueness acceptable outcome problem for games having Nash equilibrium point. A main result connecting this problem for class of games with quasi-ordered outcomes is the theorem 4. We need in the following definition.

Definition 7. An arbitrary quasi-ordered set $\langle A, \rho\rangle$ is said *satisfies (AC) condition* if every strict ascending chain of the form $a_1 \overset{\rho}{<} a_2 \overset{\rho}{<} \ldots \overset{\rho}{<} a_k \overset{\rho}{<} \ldots$ is terminated, i.e. it has a last element.

Theorem 4. *Let $G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F\rangle$ be a game with quasi-ordered outcomes and for every $i \in N$ quasi-ordered set $\langle A, \rho_i\rangle$ (AC) condition satisfies. Then there exists an unique up to the natural equivalence ε acceptable outcome if and only if game G has a special Nash equilibrium point x^0 and outcome $F\left(x^0\right)$ is a centric.*

Proof (of theorem 4). Necessity. Let a^* be an unique up to the natural equivalence ε acceptable outcome in game G. For any $i \in N$ consider the set W_i consisting of strict guaranteed outcomes of player i (see lemma 2). Denote by N_0 the set of all $i \in N$ satisfying $W_i \neq \varnothing$ and by N_1 the set of all $i \in N$ satisfying $W_i = \varnothing$. For every $i \in N_0$ fix in non-empty set W_i a maximal element a_i^* under quasi-order ρ_i (an existence of maximal element it follows from (AC) condition). Since $a_i^* \in W_i$ then there exists a strategy $x_i^* \in X_i$ such that the correlation $F\left(x_i^*, x_{N\setminus i}\right) \overset{\rho_i}{>} a_i^*$ holds for any $x_{N\setminus i} \in X_{N\setminus i}$ ($i \in N_0$). Moreover for all $i \in N_1$ fix an arbitrary strategy $x_i^* \in X_i$. Let us show that the outcome in situation $x^* = (x_i^*)$ is an acceptable one in game G. Indeed, for every $i \in N_0$ the correlation $F(x^*) \overset{\rho_i}{>} a_i^*$ holds and because a_i^* is a maximal element in subset W_i, we obtain $F(x^*) \notin W_i$, that is the outcome $F(x^*)$ is an acceptable one for every player $i \in N_0$. Since $W_i = \varnothing$ for all $i \in N_1$, any outcome of game G is acceptable for each player $i \in N_1$. Therefore the outcome $F(x^*)$ is an acceptable one for all players $i \in N$ i.e. it is an acceptable one in game G, hence in accordance with uniqueness condition we get $F(x^*) \overset{\varepsilon}{\equiv} a^*$ where ε is a natural equivalence in game G.

We affirm that x^* is Nash equilibrium point in game G. Indeed assume that in situation x^* some player $k \in N$ instead of strategy x_k^* uses another strategy $x_k \in X_k$. In accordance with definition of situation x^* we obtain that outcome $F(x^* \parallel x_k)$ remains to be acceptable for all players $i \in N$, where $i \neq k$. It is possible the following two cases.

Case 1. The outcome in situation $x^* \parallel x_k$ remains to be acceptable for player k. Then outcome $F(x^* \parallel x_k)$ is an acceptable for all players $i \in N$, hence in accordance with uniqueness condition we have $F(x^* \parallel x_k) \overset{\varepsilon}{\equiv} a^*$ and since $F(x^*) \overset{\varepsilon}{\equiv} a^*$ we obtain $F(x^* \parallel x_k) \overset{\varepsilon}{\equiv} F(x^*)$. Because $\varepsilon \subseteq \varepsilon_k \subseteq \rho_k$, in this case we have $F(x^* \parallel x_k) \overset{\rho_k}{\lesssim} F(x^*)$.

Case 2. The outcome in situation $x^* \parallel x_k$ is not acceptable for player k. Then $F(x^* \parallel x_k) \in W_k$ hence $W_k \neq \varnothing$. In accordance with (AC) condition for quasi-ordered set $\langle A, \rho_k \rangle$, the subset W_k has a maximal element $b_k^* \in W_k$ such that $F(x^* \parallel x_k) \overset{\rho_k}{\lesssim} b_k^*$. Let $x_k' \in X_k$ be a strategy of player k which strict guarantees the outcome b_k^* to him. Then $F(x^* \parallel x_k') \overset{\rho_k}{>} b_k^*$, hence, using a maximality condition for element b_k^* in subset W_k, we get $F(x^* \parallel x_k') \notin W_k$, that is outcome $F(x^* \parallel x_k')$ is an acceptable for player k. Since the outcome $F(x^* \parallel x_k')$ remains to be acceptable for other players $i \in N$ where $i \neq k$, we get that the outcome $F(x^* \parallel x_k')$ is an acceptable one in game G. Then in accordance with uniqueness condition we have $F(x^* \parallel x_k') \overset{\varepsilon}{\equiv} a^*$ where ε is a natural equivalence in game G. Thus we have the following sequence of correlations:

$$F(x^* \parallel x_k) \overset{\rho_k}{\lesssim} b_k^* \overset{\rho_k}{<} F(x^* \parallel x_k') \overset{\varepsilon}{\equiv} a^* \overset{\varepsilon}{\equiv} F(x^*).$$

Since $\varepsilon \subseteq \varepsilon_k \subseteq \rho_k$ and binary relation ρ_k satisfies the transitivity condition, we get in this case $F(x^* \parallel x_k) \overset{\rho_k}{<} F(x^*)$.

We show that situation x^* is Nash equilibrium point in game G, that is the first affirmations of theorem 4. Then other statements of theorem 4 are to be direct consequences of lemma 3. □

Corollary 3. *A game $G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F \rangle$ with quasi-ordered outcomes in which for every $i \in N$ quasi-ordered set $\langle A, \rho_i \rangle$ (AC) condition satisfies has an acceptable outcome.*

Indeed, in theorem 4, a proof of the fact that outcome $F(x^*)$ is an acceptable one in game G does not use an existence and uniqueness of an acceptable outcome condition.

4. Examples

4.1. Antagonistic games with payoff functions

Consider an antagonistic game with payoff function $\Gamma = \langle X, Y, u \rangle$ where X is a set of strategies of player 1, Y is a set of strategies of player 2, u is a payoff function. We can mean Γ a game with ordered outcomes, in which the set of strategies of players are the same, a set of outcomes is real numbers $\mathbb{R}$, realization function is the function $u(x, y)$ and preference relation is determined by the value of payoff. Put $v_1 = \sup_{x\in X} \inf_{y\in Y} u(x, y)$ be the *lower value* and $v_2 = \inf_{y\in Y} \sup_{x\in X} u(x, y)$ the *upper value* of game Γ. Consider now the following condition.

(C) If the external extremum of $\sup_{x\in X} \inf_{y\in Y} u(x, y)$ is realized at the point $x_0 \in X$ then the inner extremum of $\inf_{y\in Y} u(x_0, y)$ must be realized at some point $y_0 \in Y$.

It is easy to show that for game Γ considered as a game with ordered outcomes, the set of all acceptable outcomes for player 1 is the interval (v_1, ∞) and possibly the point v_1. Moreover, the outcome v_1 is an acceptable one for player 1 if and only if the condition (C) holds. For finding of all acceptable outcomes for player 2 we can use a dual condition (C*). Thus the set $Ac\Gamma$ consisting of all acceptable outcomes of game Γ is the interval (v_1, v_2) and possibly points v_1 and v_2. In particular let the sets X, Y be compact topological spaces and the function u is continuous on $X \times Y$. Then the conditions (C) and (C*) hold, hence in this case we have $Ac\Gamma = [v_1, v_2]$.

4.2. n-person games with payoff functions

A finding the set of acceptable outcomes in n-person game with payoff functions can be reduced to this problem for antagonistic game. Namely let $G = \langle (X_i)_{i\in N}, (u_i)_{i\in N} \rangle$ be a game of players $N = \{1, \ldots, n\}$ where X_i is a set of strategies and u_i is a payoff function of player i. We can consider G as a game with quasi-ordered outcomes in which $\mathbb{R}^N$ is a set of outcomes and for any two vectors $(y_1, \ldots, y_n), (y'_1, \ldots, y'_n) \in \mathbb{R}^N$ put

$$(y_1, \ldots, y_n) \overset{\rho_i}{\lesssim} (y'_1, \ldots, y'_n) \Leftrightarrow y_i \le y'_i.$$

Suppose in game G set of strategies of players are compact topological spaces and payoff functions are continuous on $\prod_{i\in N} X_i$. Then acceptable outcomes in game G are exactly vectors $\left(y_1^0, \ldots, y_n^0\right) \in \mathbb{R}^N$ such that for any $i \in N$ the condition $y_i^0 \ge \nu_i$ holds, where ν_i is the lower value of antagonistic game of player i against the complementary coalition $N \setminus i$.

4.3. An example of game which has not of acceptable outcomes

Consider an antagonistic game Γ_1 with payoff function given by table 1.

Table 1. Payoff function of game Γ_1

X \ Y	y_1	y_2	y_3	$\ldots$	y_n	$\ldots$	inf
x_1	1	1/2	1/3	$\ldots$	$1/n$	$\ldots$	0
x_2	2	$-1/2$	$-1/3$	$\ldots$	$-1/n$	$\ldots$	$-1/2$
sup	2	1/2	1/3	$\ldots$	$1/n$	$\ldots$	$\nu_1 = \nu_2 = \nu = 0$

In this game a set of outcomes is real numbers $\mathbb{R}$. It follows from table 1 that any outcome $r \le 0$ is not acceptable for player 1 since the strategy x_1 is an objection of player 1 to such outcome. Moreover, any outcome $r > 0$ is not acceptable for player 2: an objection of player 2 to such outcome $r > 0$ is its strategy y_n where $n = [1/r] + 1$. Therefore in game Γ_1 the set of acceptable outcomes is empty.

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