Contributions to Game Theory and Management, XI, 113–128

Game Equilibria and Transition Dynamics in Triregular Networks*

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Abstract We study game equilibria in a model of production and externalities in regular network with three types of agents who possess different productivities. Each agent may invest a part of her endowment (for instance, time or money) at the first stage; consumption at the second stage depends on her own investment and productivity as well as on the investments of her neighbors in the network. We introduce adjustment dynamics described by differential equations. We study which equilibria are possible, and which of these equilidria are dynamically stable under different combinations of parameters of the game.

Keywords: network, game equilibrium, heterogeneous agents, network formation, productivity.

1. Introduction

Social network analysis became an important research field, both as a subject area and as a methodological approach applicable to analysis of interrelations in various complex network structures, not only social, but political, economic, urban. A special place is played by the approach of network games (e.g. Bramoulle and Kranton, 2007, Galeotti et al., 2010, Goyal, 2009, Jackson, 2008, Jackson and Zenou, 2014, Martemyanov and Matveenko, 2014), which assumes that agents in network act as rational decision makers, and the profile of actions of all agents in the network is a game equilibrium. Decision of each agent is supposed to be influenced by behavior (or by knowledge) of her neighbors in the network. In majority of research on game equilibria in networks the agents are assumed to be homogeneous (except their positions in the network), and the problem is to study the relation between the agents' positions in the network and their behavior in the game equilibrium, characterized by one or another measure of centrality (e.g. Ballester et al., 2006, Bramoulle et al., 2014, Matveenko and Korolev, 2017, Naghizadeh and Liu, 2017).

However, diversity and heterogeneity have become an important aspect of contemporary social and economic life (many examples are provided by researchers of inclusiveness and social cohesion, e.g. Acemoglu and Robinson, 2012). Correspondingly, along with accounting for position of agents in the network, an important task is to account for heterogeneity of agents as a factor shaping differences in their behavior and wellbeing. This direction of research becomes actual in the literature (see e.g. Goyal, 2018).

* The research is supported by the Russian Foundation for Basic Research (project 17-06-00618).

In the present paper we add heterogeneity of agents and adjustment dynamics into a two-period consumption-investment model with network externalities (see (Romer, 1986) for a special case of complete network and (Matveenko and Korolev, 2017) for a general network case). The model considers situations in which at the first stage each agent in network, at the expense of diminishing current consumption, may make investment of some resource (such as money or time) with the goal to increase her second stage consumption. The latter depends not only on her own investment and productivity but also on investments by her neighbors in the network. Total utility of each agent depends on her consumptions at both stages. Such situations are typical for families, communities, international organizations, innovative industries, etc.

We use the concept of 'Nash equilibrium with externalities', similar to the one introduced by Romer, 1986 and Lucas, 1988. It is assumed that the agent makes her decision, being in a definite environment which is formed by herself and by her neighbors in the network. Though she participates herself in the environment, the agent in the moment of decision-making considers the environment as exogenously given.

Matveenko et al., 2017 assume the presence of two types of agents characterized by different productivities, but study only the case of complete networks. In the present paper we consider a more general class of regular (equidegree) networks with three types of agents.

We identify conditions under which an agent behaves in equilibrium in a definite way, being 'passive' (not investing), 'active' (investing a part of the available endowment) or 'hyperactive' (investing the whole endowment), study dependence of the investment on the pure externality received by the agent and the influence of the heterogeneity on the game equilibria. We introduce adjustment dynamics into the model and study dynamics of transition to the equilibrium. The dynamics pattern and the nature of the resulting equilibrium depend on the parameters characterizing heterogeneous agents. A questions studied in the paper is the enumerating of symmetric equilibria which are possible in triregular network and finding conditions under which these equilibria are possible. We study also the correlation between parameters of network, under which these equilibria are dynamically stable. We make conclusions about behavior of agents of different types after junction of regular networks; in particular, how the behavior of nonadopters (passive agents) changes when they connect to adopters (active or hyperactive) agents.

The paper is organized in the following way. The game model is formulated in Section 2. Agent's behavior in equilibrium is characterized in Section 3. Section 4 studies equilibria with heterogeneous agents in regular network of a special class. Section 5 introduces and studies the adjustment dynamics which may start after a small disturbance of initial equilibrium or after a junction of networks. Section 6 concludes.

2. The model

In a network (undirected graph) each node $i = 1, 2, ..., n$ represents an agent. At the first stage each agent i possesses initial endowment of good, e (it may be, for instance, time or money) and uses it partially for consumption at the first stage, c_1^i, and partially for investment into knowledge, k_i:

$$c_1^i + k_i = e, i = 1, 2, ..., n.$$

Investment immediately transforms one-to-one into knowledge, which is used in production of good for consumption on the second stage, c_2^i. Production in node i is described by production function:

$$F(k_i, K_i) = g_i k_i K_i,\ \ g_i > 0,$$

which depends on the state of knowledge in i-th node, k_i, and on *environment*, K_i. The environment is the sum of investments by the agent himself and her neighbors:

$$K_i = k_i + \tilde{K_i}, \tilde{K_i} = \sum_{j \in N(i)} k_i$$

where $N(i)$ – is the set of neighboring nodes of node i. The sum of investments of neighbors, $\tilde{K_i}$, will be referred as *pure externality*.

Preferences of agent i are described by quadratic utility function:

$$U_i(c_1^i, c_2^i) = c_1^i(e - ac_1^i) + d_i c_2^i,$$

where $d_i > 0$; a is a satiation coefficient. It is assumed that $c_1^i \in [0, e]$, the utility increases in c_1^i and is concave (the marginal utility decreases) with respect to c_1^i. A sufficient condition leading to the assumed property of the utility is $0 < a < 1/2$; we assume that this inequality is satisfied.

We will denote the product $d_i g_i$ by b_i and assume that $a < b_i$. Since increase of any of parameters d_i, g_i promotes increase of the second stage consumption, we will call b_i *productivity*. We will assume that $b_i \neq 2a,\ \ i = 1, 2, ..., n$. If $b_i > 2a$, we will say that i-th agent is *productive*, and if $b_i < 2a$ – that the agent is *unproductive*.

Three ways of behavior are possible: agent i is called *passive* if she makes zero investment, $k_i = 0$ (i.e. consumes the whole endowment at the first stage); *active* if $0 < k_i < e$; *hyperactive* if she makes maximally possible investment e (i.e. consumes nothing at the first stage).

Let us consider the following game. Players are the agents $i = 1, 2, ..., n$. Possible actions (strategies) of player i are values of investment k_i from the segment $[0, e]$. *Nash equilibrium with externalities* (for shortness, *equilibrium*) is a profile of actions $k_1^*, k_2^*, ..., k_n^*$, such that each k_i^* is a solution of the following problem $P(K_i)$ of maximization of i-th player's utility given environment K_i:

$$U_i(c_1^i, c_2^i) \underset{c_1^i, c_2^i, k^i}{\longrightarrow} max$$

$$\begin{cases} c_1^i = e - k^i, \\ c_2^i = F(k^i, K^i), \\ c_1^i \geq 0, c_2^i \geq 0, k^i \geq 0, \end{cases}$$

where the environment K_i is defined by the profile $k_1^*, k_2^*, ..., k_n^*$:

$$K_i = k_i^* + \sum_{j \in N(i)} k_j^*.$$

Substituting the constraints-equalities into the objective function, we obtain a new function (*payoff function*):

$$V_i(k_i, K_i) = U_i(e - k_i, F_i(k_i, K_i)) = (e - k_i)(e - a(e - K_i)) + b_i k_i K_i = \\ e^2(1-a) - k_i e(1-2a) - a k_i^2 + b_i k_i K_i. \tag{1}$$

If all players' solutions are internal ($0 < k_i^* < e, i = 1, 2, ..., n$), i.e. all players are active, the equilibrium will be referred to as *inner* equilibrium. Clearly, the inner equilibrium (if it exists for given values of parameters) is defined by the system

$$D_1 V_i(k_i, K_i) = 0, \; i = 1, 2, ..., n. \tag{2}$$

Here

$$D_1 V_i(k_i, K_i) = e(2a - 1) - 2a k_i + b_i K_i. \tag{3}$$

3. Indication of agent's ways of behavior

We will denote by k_i^S the root of the equation

$$D_1 V_i(k_i, K_i) = (b_i - 2a)k_i + b_i \tilde{K}_i - e(1 - 2a) = 0.$$

Thus,

$$k_i^S = \tfrac{e(2a-1)+b_i \tilde{K}_i}{2a-b_i},$$

where $\tilde{K}_i$ is the pure externality received by the agent.

Remark 1. Lemma 2.1 in (Matveenko et al., 2017) and Corollary 2.1 in (Matveenko et al., 2017) give us a practical method of examination, whether the collection of agents' investments is in equilibrium. Namely, the collection $k_1, k_2, \ldots, k_n$ may be in equilibrium only if for every agent i, $i = 1, 2, \ldots, n$

1) If $k_i = 0$, then $\tilde{K}_i \leq \frac{e(1-2a)}{b_i}$;
2) If $0 < k_i < e$, then $k_i = k_i^S$;
3) If $k_i = e$, then $\tilde{K}_i \geq \frac{e(1-b_i)}{b_i}$.

Lemma 1 (Lemma 2.2 in (Matveenko et al., 2017)). .
In equilibrium i-th agent is passive iff

$$K_i \leq \frac{e(1-2a)}{b_i}; \tag{4}$$

i-th agent is active iff

$$\frac{e(1-2a)}{b_i} < K_i < \frac{e}{b_i}; \tag{5}$$

i-th agent is hyperactive iff

$$K_i \geq \frac{e}{b_i}. \tag{6}$$

Remark 2. In any network, in which all agents have the same environment, there cannot be equilibrium in which an agent with a higher productivity is active while an agent with a lower productivity is hyperactive, or when an agent with a higher productivity is passive while an agent with a lower productivity is active or hyperactive.

4. Equilibria in triregular network with heterogeneous agents

Definition 1. Let the set of nodes $1, 2, \ldots, n$ be decomposed into disjoint classes in such way that any nodes belonging the same class have the same productivity and the same numbers of neighbors from each class. The classes will be referred as *types* of nodes. Type i is characterized by productivity b_i and by vector $\boldsymbol{t}_i = (t_{i1}, t_{i2}, \ldots, t_{ik})$, where t_{ij} is the number of neighbors of type j for any node of type i.

Let us describe an algorithm of subdivision of the set of nodes of network into types. Let s be a current number of subsets of subdivision. Initially s is the number of various productivities.

Iteration of the algorithm. Consider nodes of the first subset. If all of them have the same numbers of neighbors in each subset $1, 2, ..., s$, then the first subset is not changed. In the opposite case, we divide the first subset into new subsets in such way that all nodes of each new subset have the same vector of numbers of neighbors in subsets.

We proceed in precisely same way with the second, the third, ..., the s-th subset. If on the present iteration the number of subsets s have not changed, then the algorithm finishes its work. If s has increased, then the new iteration is executed.

The number of subsets s does not decrease in process of the algorithm. Since s is bounded from above by the number of nodes in the network, the algorithm converges. It is clear that the algorithm divides the set of nodes into the minimal possible number of classes.

Definition 2. A network in which each node has the same degree (number of neighbors) is referred as *regular*.

Definition 3. Let us consider a regular network consisting of three types of agents with productivities b_i and vectors $t_i = (t_{i1}, t_{i2}, t_{i3})$, $i = 1, 2, 3$; $b_1 > b_2 > b_3$. Let the following conditions be satisfied:

$$t_{11} + 1 = t_{21} = t_{31} = n_1,$$

$$t_{12} = t_{22} + 1 = t_{32} = n_2,$$

$$t_{13} = t_{23} = t_{33} + 1 = n_3.$$

Such network will be referred as *triregular*.

The triregularity seems to be a natural specification of regularity. In triregular network, any agent has n_i links with type i $(i = 1, 2, 3)$. (Since each agent of type i is "linked" in some sense with herself, she has only $(n_i - 1)$ links with other agents of type i).

A special case of triregular network is a complete network with $n_1 + n_2 + n_3$ nodes which is received in result of junction of three complete networks with n_1, n_2 and n_3 nodes.

Definition 4. Equilibrium (or any other situation) is called *symmetric*, if all players of the same type choose the same action (make the same investment).

Let a triregular network be in a symmetric equilibrium, in which each i^{th} type agent makes investment k_i, $i = 1, 2, 3$. Then, for each agent environment is equal to $K = k_1 n_1 + k_2 n_2 + k_3 n_3$. According to Remark 2, only 10 symmetric equilibria are possible. The following proposition lists these possible symmetric equilibria and provides conditions of their existence.

Proposition 1. *In triregular network the following symmetric equilibria exist.*

1) Equilibrium with all hyperactive agents (HHH) is possible iff

$$b_1 > b_2 > b_3 \geq \frac{1}{n_1 + n_2 + n_3}. \tag{7}$$

2) Equilibrium in which 1^{st} and 2^{nd} types agents are hyperactive and 3^{rd} type agents are active (HHA) is possible if

$$0 < \frac{1 - 2a - (n_1 + n_2)b_3}{n_3 b_3 - 2a} < 1, \tag{8}$$

$$n_1 + n_2 + n_3 \frac{1 - 2a - (n_1 + n_2)b_3}{n_3 b_3 - 2a} \geq \frac{1}{b_2}. \tag{9}$$

3) Equilibrium in which 1^{st} type agents are hyperactive and 2^{nd} and 3^{rd} types agents are active (HAA) is possible if

$$0 < \frac{e(1 - 2a)[(b_3 - b_2)n_3 - 2a] + 2aen_1 b_2}{2a[2a - (n_2 b_2 + n_3 b_3)]} < 1, \tag{10}$$

$$0 < \frac{e(1 - 2a)[(b_2 - b_3)n_2 - 2a] + 2aen_1 b_3}{2a[2a - (n_2 b_2 + n_3 b_3)]} < 1, \tag{11}$$

$$n_1 + n_2 \frac{e(1 - 2a)[(b_3 - b_2)n_3 - 2a] + 2aen_1 b_2}{2a[2a - (n_2 b_2 + n_3 b_3)]} +$$

$$n_3 \frac{e(1 - 2a)[(b_2 - b_3)n_2 - 2a] + 2aen_1 b_3}{2a[2a - (n_2 b_2 + n_3 b_3)]} \geq \frac{1}{b_1}. \tag{12}$$

4) Equilibrium in which 1^{st} and 2^{nd} types agents are hyperactive and 3^{rd} type agents are passive (HHP) is possible iff

$$b_1 > b_2 \geq \frac{1}{n_1 + n_2}, \quad b_3 \leq \frac{1 - 2a}{n_1 + n_2}. \tag{13}$$

5) Equilibrium in which 1^{st} type agents are hyperactive, 2^{nd} type agents are active and 3^{rd} type agents are passive (HAP) is possible if

$$0 < \frac{1 - 2a - n_1 b_2}{n_2 b_2 - 2a} < 1, \tag{14}$$

$$\frac{1}{b_1} \leq n_1 + n_2 \frac{1 - 2a - n_1 b_2}{n_2 b_2 - 2a} \leq \frac{1 - 2a}{b_3}. \tag{15}$$

6) Equilibrium in which 1^{st} type agents are hyperactive and 2^{nd} and 3^{rd} types agents are passive (HPP) is possible iff

$$b_1 \geq \frac{1}{n_1}, \quad b_3 < b_2 \leq \frac{1-2a}{n_1}. \tag{16}$$

7) Equilibrium in which agents of all types are active (AAA) exists if

$$0 < \frac{(1-2a)\big((n_2+n_3)b_1 - n_2b_2 - n_3b_3 + 2a\big)}{2a\big((n_1b_1+n_2b_2+n_3b_3)-2a\big)} < 1, \tag{17}$$

$$0 < \frac{(1-2a)\big((n_1+n_3)b_2 - n_1b_1 - n_3b_3 + 2a\big)}{2a\big((n_1b_1+n_2b_2+n_3b_3)-2a\big)} < 1, \tag{18}$$

$$0 < \frac{(1-2a)\big((n_1+n_2)b_3 - n_1b_1 - n_2b_2 + 2a\big)}{2a\big((n_1b_1+n_2b_2+n_3b_3)-2a\big)} < 1. \tag{19}$$

8) Equilibrium in which 1^{st} type agents and 2^{nd} type agents are active and 3^{rd} type agents are passive (AAP) is possible if

$$0 < \frac{(1-2a)\big(n_2(b_2-b_1)-2a\big)}{2a(2a-n_1b_1-n_2b_2)} < 1, \tag{20}$$

$$0 < \frac{(1-2a)\big(n_1(b_1-b_2)-2a\big)}{2a(2a-n_1b_1-n_2b_2)} < 1, \tag{21}$$

$$n_1\frac{(1-2a)\big(n_2(b_2-b_1)-2a\big)}{2a(2a-n_1b_1-n_2b_2)} + n_2\frac{(1-2a)\big(n_1(b_1-b_2)-2a\big)}{2a(2a-n_1b_1-n_2b_2)} \leq \frac{1-2a}{b_3}. \tag{22}$$

9) Equilibrium in which 1^{st} type agents are active and 2^{nd} type agents and 3^{rd} type agents are passive (APP) is possible if

$$b_1 > \frac{1}{n_1}, \tag{23}$$

$$\frac{n_1(1-2a)}{n_1b_1-2a} \leq \frac{e(1-2a)}{b_2} < \frac{e(1-2a)}{b_3}. \tag{24}$$

10) Equilibrium with all passive agents (PPP) is always possible.

Proof. 1) Follows from Lemma 1.

2) This equilibrium is possible if inequality (8) is checked, because equation (2) for 3^{rd} type agents is

$$n_1b_3e + n_2b_3e + (n_3b_3-2a)k_3 = e(1-2a).$$

According to (6), the equilibrium exists under (9).

3) The system of equations (2) for 2^{nd} and 3^{rd} types agents is

$$\begin{cases} n_1b_2e + (n_2b_2-2a)k_2 + n_3b_2k_3 = e(1-2a), \\ n_1b_3e + n_2b_3k_2 + (n_3b_3-2a)k_3 = e(1-2a). \end{cases}$$

The solution of this system is

$$k_2^S = \frac{e(1-2a)[(b_3-b_2)n_3-2a]+2aen_1b_2}{2a[2a-(n_2b_2+n_3b_3)]},$$

$$k_3^S = \frac{e(1-2a)[(b_2-b_3)n_2-2a]+2aen_1b_3}{2a[2a-(n_2b_2+n_3b_3)]}.$$

That implies the conditions (10) and (11). From (6) follows the condition (12).

4) Since in this case the environment is $K=(n_1+n_2)e$, according to (4) and (6), the equilibrium exists iff (13) is checked.

5) The equation (2) for 2^{nd} type agents is

$$n_1b_2e+(n_2b_2-2a)k_2=e(1-2a),$$

that implies the condition (14). From (4) and (6) follows condition (15).

6) According to (4) and (6), the equilibrium is possible iff (16) holds.

7) The system of equations (2) turns into

$$\begin{cases}(n_1b_1-2a)k_1+n_2b_1k_2+n_3b_1k_3=e(1-2a),\\ n_1b_2k_1+(n_2b_2-2a)k_2+n_3b_2k_3=e(1-2a),\\ n_1b_3k_1+n_2b_3k_2+(n_3b_3-2a)k_3=e(1-2a).\end{cases}$$

We solve this system by Kramer method and obtain

$$k_1^S = \frac{e(1-2a)\big((n_2+n_3)b_1-n_2b_2-n_3b_3+2a\big)}{2a\big((n_1b_1+n_2b_2+n_3b_3)-2a\big)},$$

$$k_2^S = \frac{e(1-2a)\big((n_1+n_3)b_2-n_1b_1-n_3b_3+2a\big)}{2a\big((n_1b_1+n_2b_2+n_3b_3)-2a\big)},$$

$$k_3^S = \frac{e(1-2a)\big((n_1+n_2)b_3-n_1b_1-n_2b_2+2a\big)}{2a\big((n_1b_1+n_2b_2+n_3b_3)-2a\big)}.$$

Hence, the necessary and sufficient conditions of existence of the inner equilibrium are (17), (18), (19). Under these inequalities, the inner equilibrium is

$$k_1=k_1^S,\quad k_2=k_2^S,\quad k_3=k_3^S.$$

8) The system of equations (2) for 1^{st} and 2^{nd} types agents in this case is

$$\begin{cases}(n_1b_1-2a)k_1+b_1n_2k_2=e(1-2a),\\ n_1b_2k_1+(n_2b_2-2a)k_2=e(1-2a).\end{cases}$$

The solution of this system by Kramer method is

$$k_1^S = \frac{e(1-2a)\big(n_2(b_2-b_1)-2a\big)}{2a(2a-n_1b_1-n_2b_2)},$$

$$k_2^S = \frac{e(1-2a)\big(n_1(b_1-b_2)-2a\big)}{2a(2a-n_1b_1-n_2b_2)}.$$

That implies the conditions (20) and (21). According to (4), the equilibrium is possible if (22).

9) This equilibrium is possible if inequality

$$0 < \frac{1-2a}{n_1 b_1 - 2a} < 1 \tag{25}$$

is checked, because equation (2) for 1st type agents is

$$(n_1 b_1 - 2a)k_1 = e(1-2a).$$

But (25) is equivalent to (23). According to (4), the equilibrium exists under (24).

10) Follows from Lemma 1.

□

5. Adjustment dynamics and dynamic stability of equilibria

Now we introduce adjustment dynamics which may start after a small deviation from equilibrium or after junction of networks each of which was initially in equilibrium. We model the adjustment dynamics in the following way.

Definition 5. In the adjustment process, each agent maximizes her utility by choosing a level of her investment; at the moment of decision-making she considers her environment as exogenously given. Correspondingly, if $k_i(t_0) = 0$, where t_0 is an arbitrary moment of time, and $D_1V_i(k_i, K_i)|_{k_i=0} \le 0$, then $k_i(t) = 0$ for any $t > t_0$, and if $k_i(t_0) = e$ and $D_1V_i(k_i, K_i)|_{k_i=e} \ge 0$, then $k_i(t) = e$ for any $t > t_0$; in all other cases, $k_i(t)$ satisfies the differential equation:

$$\dot{k_i} = \frac{b_i}{2a}\tilde{K}_i + \frac{b_i - 2a}{2a}k_i - \frac{e(1-2a)}{2a}. \tag{26}$$

Definition 6. The equilibrium is called *dynamically stable* if, after a small deviation of one of the agents from the equilibrium, dynamics starts which returns the equilibrium back to the initial state. In the opposite case, the equilibrium is called *dynamically unstable.*

In triregular network, let in initial time period each i-th type agent invest k_{0i} $(i = 1,2,3)$. Correspondingly, the environment (common for all agents) in the initial period is $K = n_1k_{01} + n_2k_{02} + n_3k_{03}$.

Assume that for each i $(i = 1,2,3)$ either $k_{0i} = 0$ and $D_1V_1(k_i, K)|_{k_i=0} > 0$, or $k_{0i} = e$ and $D_1V_1(k_i, K)|_{k_i=e} < 0$, or $k_{0i} \in (0, e)$. Then Definition 4.1 implies that the dynamics is described by the system of differential equations.

$$\begin{cases} \dot{k_1} = \frac{n_1b_1-2a}{2a}k_1 + \frac{n_2b_1}{2a}k_2 + \frac{n_3b_1}{2a}k_3 + \frac{e(2a-1)}{2a}, \\ \dot{k_2} = \frac{n_1b_2}{2a}k_1 + \frac{n_2b_2-2a}{2a}k_2 + \frac{n_3b_2}{2a}k_3 + \frac{e(2a-1)}{2a}, \\ \dot{k_3} = \frac{n_1b_3}{2a}k_1 + \frac{n_2b_3}{2a}k_2 + \frac{n_3b_3-2a}{2a}k_3 + \frac{e(2a-1)}{2a} \end{cases} \tag{27}$$

with initial conditions

$$k_i^0 = k_{0i}, \quad i = 1,2,3. \tag{28}$$

Proposition 2. *The general solution of the system of differential equations* (27) *has the form*

$$k(t) = C_1 \cdot exp\{-t\} \begin{pmatrix} -n_3 \\ 0 \\ n_1 \end{pmatrix} + C_2 \cdot exp\{-t\} \begin{pmatrix} 0 \\ -n_3 \\ n_2 \end{pmatrix} +$$

$$+C_3 \cdot exp\Big\{ \Big(\frac{n_1 b_1 + n_2 b_2 + n_3 b_3}{2a} - 1 \Big) t \Big\} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}, \tag{29}$$

where $(D_1, D_2, D_3)^T$ *is the steady state of* (27),

$$D_i = \frac{e(1-2a)\big[(n_1+n_2+n_3)b_i - (n_1 b_1 + n_2 b_2 + n_3 b_3) + 2a\big]}{2a\big(n_1 b_1 + n_2 b_2 + n_3 b_3 - 2a\big)}, i = 1, 2, 3. \tag{30}$$

The solution of the Cauchy differential problem (27)–(28) *has the form*

$$k(t) = \frac{k^0 - \tilde{D} + (D_1 - k_1^0)(n_1 b_1 + n_2 b_2 + n_3 b_3) n_3}{(n_1 b_1 + n_2 b_2 + n_3 b_3) n_3} \cdot exp\{-t\} \begin{pmatrix} -n_3 \\ 0 \\ n_1 \end{pmatrix} +$$

$$+\frac{k^0 - \tilde{D} + (D_2 - k_2^0)(n_1 b_1 + n_2 b_2 + n_3 b_3) n_3}{(n_1 b_1 + n_2 b_2 + n_3 b_3) n_3} \cdot exp\{-t\} \begin{pmatrix} 0 \\ -n_3 \\ n_2 \end{pmatrix} +$$

$$+\frac{k^0 - \tilde{D}}{n_1 b_1 + n_2 b_2 + n_3 b_3} \cdot exp\Big\{ \Big(\frac{n_1 b_1 + n_2 b_2 + n_3 b_3}{2a} - 1 \Big) t \Big\} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} +$$

$$+ \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}, \tag{31}$$

where

$$\tilde{D} = n_1 D_1 + n_2 D_2 + n_3 D_3 = \frac{e(1-2a)(n_1+n_2+n_3)}{n_1 b_1 + n_2 b_2 + n_3 b_3 - 2a}, \tag{32}$$

$$k^0 = n_1 k_1^0 + n_2 k_2^0 + n_3 k_3^0. \tag{33}$$

Proof. The characteristic equation of system (27) is

$$\begin{vmatrix} \frac{n_1 b_1}{2a} - (\lambda+1) & \frac{n_2 b_1}{2a} & \frac{n_3 b_1}{2a} \\ \frac{n_1 b_2}{2a} & \frac{n_2 b_2}{2a} - (\lambda+1) & \frac{n_3 b_2}{2a} \\ \frac{n_1 b_3}{2a} & \frac{n_2 b_3}{2a} & \frac{n_3 b_3}{2a} - (\lambda+1) \end{vmatrix} =$$

$$= (\lambda+1)^2 \Big(\frac{n_1 b_1}{2a} + \frac{n_2 b_2}{2a} + \frac{n_3 b_3}{2a} \Big) + (\lambda+1)^3 = 0.$$

Thus, the eigenvalues are

$$\lambda_{1,2} = -1, \quad \lambda_3 = \frac{n_1 b_1 + n_2 b_2 + n_3 b_3}{2a} - 1.$$

Eigenvectors corresponding $\lambda = -1$ are

$$e_1 = \begin{pmatrix} -n_3 \\ 0 \\ n_1 \end{pmatrix}$$

and

$$e_2 = \begin{pmatrix} 0 \\ -n_3 \\ n_2 \end{pmatrix},$$

while an eigenvector corresponding λ_3 can be found as a solution of the system of equations

$$\begin{cases} -(n_2 b_2 + n_3 b_3)x_1 + n_2 b_1 x_2 + n_3 b_1 x_3 = 0, \\ n_1 b_2 x_1 - (n_1 b_1 + n_3 b_3)x_2 + n_3 b_2 x_3 = 0, \\ n_1 b_3 x_1 + n_2 b_3 x_2 - (n_1 b_1 + n_2 b_2)x_3 = 0. \end{cases}$$

We find

$$e_2 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The general solution of the homogeneous system of differential equations corresponding (27) has the form

$$(k(t))_g = C_1 \cdot exp\{-t\} \begin{pmatrix} -n_3 \\ 0 \\ n_1 \end{pmatrix} + C_2 \cdot exp\{-t\} \begin{pmatrix} 0 \\ -n_3 \\ n_2 \end{pmatrix} +$$

$$+C_3 \cdot exp\Big\{\Big(\frac{n_1 b_1 + n_2 b_2 + n_3 b_3}{2a} - 1\Big)t\Big\} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

As a partial solution of the system (27) we take its steady state, i.e. the solution of the linear system

$$(n_1 x_1 + n_2 x_2 + n_3 x_3)b_i - 2a x_i = e(1 - 2a), \quad i = 1, 2, 3.$$

The solution is (30); hence, the general solution of the system (27) has the form (29). In solution of the Cauchy problem (27)–(28), constants of integration are defined from the initial conditions:

$$\begin{pmatrix} k_1^0 \\ k_2^0 \\ k_3^0 \end{pmatrix} = C_1 \begin{pmatrix} -n_3 \\ 0 \\ n_1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -n_3 \\ n_2 \end{pmatrix} + C_3 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}. \tag{34}$$

Multiplying the first scalar equation of system (34) by n_1, the second equation by n_2 and the third equation by n_3 and adding we obtain

$$C_3 = \frac{k^0 - \tilde{D}}{n_1 b_1 + n_2 b_2 + n_3 b_3}.$$

Substituting this expression in the first and the second equations of system (34) we obtain

$$C_1 = \frac{k_0 - \tilde{D} + (D_1 - k_1^0)(n_1 b_1 + n_2 b_2 + n_3 b_3) n_3}{n_1 b_1 + n_2 b_2 + n_3 b_3) n_3},$$

$$C_2 = \frac{k_0 - \tilde{D} + (D_2 - k_2^0)(n_1 b_1 + n_2 b_2 + n_3 b_3) n_3}{n_1 b_1 + n_2 b_2 + n_3 b_3) n_3}.$$

Substituting for C_1, C_2 and C_3 into (29) we obtain (31). □

Proposition 3. *The conditions of dynamic stability/instability of the equilibria listed in Proposition 1, if they exist, are the following.*

1. The equilibrium with all hyperactive agents (HHH) is stable iff

$$b_1 > b_2 > b_3 > \frac{1}{n_1 + n_2 + n_3}. \tag{35}$$

2. Equilibrium in which 1st and 2nd types agents are hyperactive and 3rd type agents are active (HHA) is stable iff $n_3 = 1$ *and*

$$\frac{1 - 2a}{n_1 + n_2} < b_3 < \frac{1}{n_1 + n_2 + 1}, \tag{36}$$

$$n_1 + n_2 + \frac{1 - 2a - (n_1 + n_2) b_3}{b_3 - 2a} > \frac{1}{b_2} > \frac{1}{b_1}. \tag{37}$$

3. Equilibrium in which 1st type agents are hyperactive and 2nd and 3rd types agents are active (HAA) is unstable.

4. Equilibrium in which 1st and 2nd types agents are hyperactive and 3rd type agents are passive (HHP) is stable iff

$$b_1 > b_2 > \frac{1}{n_1 + n_2}, \quad b_3 < \frac{1 - 2a}{n_1 + n_2}. \tag{38}$$

5. Equilibrium in which 1st type agents are hyperactive, 2nd type agents are active and 3rd type agents are passive (HAP) is stable if $n_2 = 1$ *and*

$$\frac{1 - 2a}{n_1} < b_2 < \frac{1}{n_1 + 1}, \tag{39}$$

$$\frac{1}{b_1} < n_1 + \frac{1 - 2a - n_1 b_2}{b_2 - 2a} < \frac{1 - 2a}{b_3}. \tag{40}$$

6. Equilibrium in which 1st type agents are hyperactive and 2nd and 3rd types agents are passive (HPP) is stable iff

$$b_1 > \frac{1}{n_1}, \quad b_3 < b_2 < \frac{1 - 2a}{n_1}. \tag{41}$$

7. The equilibrium in which agents of all types are active (AAA) is always unstable.

8. Equilibrium in which 1st type and 2nd type agents are active and 3rd type agents are passive (AAP) is unstable.

9. Equilibrium in which 1st type agents are active and 2nd type and 3rd type agents are passive (APP) is unstable.

10. The equilibrium in which agents of all types are passive (PPP) is always stable.

Proof. 1. According to equation (3), in the point (e, e, e) we have

$$D_1V_i(k_i, K)|_{k_1=k_2=k_3=e} = b_i(n_1 + n_2 + n_3)e - e, \quad i = 1, 2, 3.$$

All the three derivatives are positive iff (35) is checked; hence, according to Definition 5 this equilibrium is stable.

2. The differential equation (26) for 3rd type agents in this case turns into

$$\dot{k}_3 = \frac{e(2a-1) + e(n_1+n_2)b_3}{2a} + \frac{n_3b_3 - 2a}{2a}k_3.$$

Thus this equilibrium may be stable only if $n_3 = 1$, $b_3 < 2a$. But in this case the condition of 3rd type agents activity (8) holds iff (36) holds. This condition implies

$$2a(n_1 + n_2 + 1) > 1;$$

thus, (36) involves the condition $b_3 < 2a$.

According to equations (3), (9),

$$D_1V_1(k_1, K)|_{k_1=k_2=k_3=e} = b_1\left(n_1 + n_2 + n_3\frac{1 - 2a - (n_1+n_2)b_3}{n_3b_3 - 2a}\right)e - e \geq 0,$$
$$D_1V_2(k_2, K)|_{k_1=k_2=k_3=e} = b_2\left(n_1 + n_2 + n_3\frac{1 - 2a - (n_1+n_2)b_3}{n_3b_3 - 2a}\right)e - e \geq 0.$$

However, for dynamic stability, according to Definition 5, the strict inequality (37) is needed.

3. The system of differential equations (27) for agents of 2nd and 3rd types in this case turns into

$$\begin{cases} \dot{k}_2 = \frac{n_1b_2e}{2a} + \frac{n_2b_2-2a}{2a}k_2 + \frac{n_3b_2}{2a}k_3 + \frac{e(2a-1)}{2a}, \\ \dot{k}_3 = \frac{n_1b_3e}{2a} + \frac{n_2b_3}{2a}k_2 + \frac{n_3b_3-2a}{2a}k_3 + \frac{e(2a-1)}{2a}. \end{cases}$$

The eigenvalues of this system are

$$\lambda_1 = -1,$$
$$\lambda_2 = -1 + \frac{n_2b_2 + n_3b_3}{2a} > 0.$$

Thus the system is unstable.

4. According to Definition 5 and equation (3), the necessary and sufficient conditions for stability are

$$D_1V_1(k_1,K)|_{k_1=e,k_2=e,k_3=0} = b_1(n_1+n_2)e - e > 0,$$
$$D_1V_2(k_2,K)|_{k_1=e,k_2=e,k_3=0} = b_2(n_1+n_2)e - e > 0,$$
$$D_1V_3(k_3,K)|_{k_1=e,k_2=e,k_3=0} = b_3(n_1+n_2)e - e(1-2a) < 0,$$

which are equivalent to (38).

5. The differential equation (26) for 2nd type agents in this case turns into

$$\dot{k}_2 = \frac{e(2a-1)+en_1b_2}{2a} + \frac{n_2b_2-2a}{2a}k_2.$$

Thus this equilibrium may be stable only if $n_2 = 1$, $b_2 < 2a$. But in this case the condition of 3rd type agents activity (14) holds iff (39) holds. This equation implies

$$2a(n_1+1) > 1,$$

thus (39) involves the condition $b_2 < 2a$. For the stability of equilibrium the condition (15) of hyperactivity of 1st type agents and passivity of 3rd type agents shall hold as strict inequality (40), according to Definition 5.

6. According to Definition 5, for stability of this equilibrium the condition (16) of its existence shall hold as strict inequalities (41).

7. One of the eigenvalues of the system (27) is

$$\lambda_3 = \frac{n_1b_1+n_2b_2+n_3b_3}{2a} - 1 > 0;$$

hence, the equilibrium is unstable.

8. The system of differential equations (27) for agents of 1st and 2nd types in this case turns into

$$\begin{cases} \dot{k}_1 = \frac{n_1b_1-2a}{2a}k_1 + \frac{n_2b_1}{2a}k_2 + \frac{e(2a-1)}{2a}, \\ \dot{k}_2 = \frac{n_1b_2}{2a}k_1 + \frac{n_2b_2-2a}{2a}k_2 + \frac{e(2a-1)}{2a}. \end{cases}$$

The eigenvalues of this system are

$$\lambda_1 = -1,$$
$$\lambda_2 = -1 + \frac{n_1b_1+n_2b_2}{2a} > 0.$$

Thus, the system is unstable.

9. The differential equation (26) for 1st type agents in this case turns into

$$\dot{k}_1 = \frac{b_1n_1-2a}{2a}k_1 + \frac{e(2a-1)}{2a}.$$

This equilibrium may be stable only if $n_1 = 1$, $b_1 < 2a$. But in this case the condition of 1st type agents activity (25) is wrong. Thus, this equilibrium is always unstable.

10. According to equation (3), we have

$$D_1V_1(k_i,K)|_{k_1=k_2=k_3=0} = e(2a-1) < 0, \quad i = 1,2,3.$$

According to Definition 5, this equilibrium is stable.

□

From this Proposition follows, in particular, that if three regular networks with productivities b_1, b_2 and b_3 ($b_1 > b_2 > b_3$), being initially in equilibria with hyperactive agents of 1^{st} and 2^{nd} networks and with passive agents of 3^{rd} network, unify, then under condition

$$b_3 < \frac{1-2a}{n_1+n_2}$$

(i.e. if productivity of the passive agents is sufficiently low), in united network shall be no transition process. The state $\{k_1 = e, k_2 = e, k_3 = 0\}$ shall be a stable equilibrium. But if before the unification the agents of the 1^{st} network were passive, but the agents of the 2^{nd} and of the 3^{rd} networks were hyperactive, in the united network arises a transition process. As result of this process, the unified network shall come in stable equilibrium in which all the agents are hyperactive.

Agents, who are initially active in a symmetric equilibrium in regular network (which implies that their productivities are sufficiently high), also may increase their level of investment in result of unification with other regular networks with hyperactive or active agents. The unified network comes into equilibrium in which all agents are hyperactive.

6. Conclusion

Research on the role of heterogeneity of agents in social and economic networks is rather new in the literature. In our model we assume presence of three types of agents possessing different productivities. At the first stage each agent in network may invest some resource (such as money or time) to increase her gain at the second stage. The gain depends on her own investment and productivity, as well as on investments of her neighbors in the network. Such situations are typical for various social, economic, political and organizational systems. In framework of the model, we consider relations between network structure, incentives, and agents' behavior in the game equilibrium state in terms of welfare (utility) of the agents.

We introduce adjustment dynamics which may start after a deviation from equilibrium or after a junction of networks initially being in equilibrium. Earlier, a special case of complete networks was considered in (Matveenko et al., 2017). Here we introduce a more general case of triregular networks and study behavior of agents with different productivities. In triregular networks we enumerate all the equilibria, which are possible under certain conditions. We find also the conditions under which these equilibria are dynamically stable.

A natural task for future research is to expand the results to broader classes of networks.

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