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Dynamic Nash Bargaining Solution for Two-stage Network Games

Jie Junnan

St. Petersburg State University
7/9 Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail: junnanjie@gmail.com

Abstract In this paper, two-stage network games are studied. At first stage of the game players form a network, while at second stage they choose strategies according to the network realized at the first stage. However, there are two kinds of two-stage networks. The first is a special class of two-stage network games when players have the opportunity to revised their network which they formed before. And the second is classical two-stage network. Cooperative setting is considered. In the cooperative case, we use Nash Bargaining Solution as a solution concept. It is demonstrated that the Nash Bargaining Solution satisfies the time consistency property for the special class of two-stage network game. But its not true for a classical two-stage network game.

Keywords: network, time-consistency, Nash Bargaining solution.

1. Model

Consider the model in details. Let $N = \{1, ..., n\}$ be a finite set of players who can interact with each other. The interaction between two players means the existence of a link connecting them and,therefore,communication between them. On the contrary, the absence of a link connecting players means the absence of any communication between them. Under these assumptions cooperation of players is said to be restricted by a communication structure (or a network). A pair (N, g) is called a network, where N is a set of nodes (it coincides with the set of players), and $g \in N \times N$ is a set of links. If a pair $(i, j) \in g$, there is a link connecting players i and j, and, therefore, generating communication of players in network. Below to simplify notations, the network will be identified with a set of its links and denoted by g, and a link (i, j) in the network will be denoted by ij. All links are non-directed, so $ij = ji$. The two stage network game under consideration we denote as G.

2. First stage: network formation

Having the player set N given, define the link formation rule in a standard way: links are formed as a result of players' simultaneous choices. Let $M_i \subseteq N/\{i\}$ be the set of players whom player $i \in N$ can offer a mutual link, and $a_i \in \{0, ..., n-1\}$ be the maximal number of links which player i can maintain (and, therefore, can offer). Behavior of player $i \in N$ at the first stage is an n-dimensional profile $g_i = (g_{i1}, ..., g_{in})$ whose entries are defined as:

$$g_{ij} = \begin{cases} 1, \text{ if player i offer a link to j } \in M_i \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

subject to the constraint:

$$\sum_{j \in N} g_{ij} \leq a_i. \tag{2}$$

The condition $g_{ij} = 0, i \in N$ excludes loops from the network, whereas (2) shows that the number of possible links is limited. If $M_i = N/\{i\}$, player i can offer a link to any player, whereas if $a_i = n-1$, he can maintain any number of links. A set of all possible behaviors of player $i \in N$ at the first stage satisfying (1), (2) is denoted by G_i. The Cartesian product $\prod_{i \in N} G_i$ is the set of behavior profiles at the first stage. It is supposed that players choose their behaviors at the first stage simultaneously and independently of each other. In particular, player $i \in N$ choose $g_i \in G_i$, and as a result the behavior profile $g_i = (g_{i1}, ..., g_{in})$ is formed. Under the above assumptions, an undirected link $ij = ji$ is established in network g if and only if $g_{ij} = g_{ji}$, g consists of mutual links which were offered only by both players.

3. Second stage: choosing controls

Denote the game on the second stage over the network g, as $\Gamma(g)$. Having formed the network g, players choose their behaviors at the second stage. Define neighbors of player i in the network g as elements of set $N_i(g) = \{j \in N \setminus \{i\} : ij \in g\}$. Players are allowed to reconsider their decisions made at the first stage by giving them the opportunity to break the previous selected links. Define components of an n-dimensional profile $d_i(g)$ as follows:

$$d_{ij} = \begin{cases} 1, & \text{player } i \text{ doesn't break the link formed with player } j \in N_i(g). \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Elements $d_i(g)$ satisfying (2), (3) are denoted by $D_i(g), i \in N$. It is obvious that profile $(d_1(g) \dots d_n(g))$ applied to the network g changes its structure and forms a new network,denoted by g^d. Network g^d is obtained from g by removing links ij such that either $d_{ij}(g) = 0$ or $d_{ji}(g) = 0$.

Moreover, at the second stage player $i \in N$ chooses control u_i form a finite set U_i. Then, behavior of player $i \in N$ at the second stage is a pair $(d_i(g), u_i)$: it defines, on the one hand, links to be removed $(d_i(g)$, and, on the other hand, control u_i.

A payoff function K_i of player $i \in N$ depends on both new network g^d and controls $u_i, i \in N$. Specifically, it depends on player i's behavior at the second stage as well as behavior of his neighbors in the network g^d. i.e., $K_i\left(u_i, u_{N_i(g^d)}\right)$ is a nonnegative real-valued function defined on $U_i \times \prod_{j \in N_i(g^d)} U_j$. Here $u_{N_i(g^d)}$ denotes a profile of controls u_j chosen by all neighbors $j \in N_i(g^d)$ of players i in the network g^d. Assume that functions $K_i, i \in N$, satisfy the following property:

(P): For any two networks g and g' s.t. $g' \subseteq g$, controls $(u_i, u_{N_i(g^d)}) \in U_i \times \prod_{j \in N_i(g^d)} U_j$, and players i, the inequality $|N_i(g)| \geq |N_i(g')|$ implies the inequality $K_i\left(u_i, u_{N_i(g)}\right) \geq K_i\left(u_i, u_{N_i(g')}\right)$. Also we suppose that the payoff of an isolated player is equal to 0.

3.1. Introduce the definition of Nash Bargaining solution

Let K be the set of all possible payoffs in the game. Denote v_i the lower value of the zero-sum game between player i and player N/i, with the payoff of player i equal to K_i. Consider the following expression:

$$\max_{K_1 \geq v_1, K_i \geq v_i, \ldots, K_n \geq v_n; (K_1, K_2, \ldots, K_n) \in K.} (K_1 - v_1) \ldots (K_n - v_n) = \\ = \left(\bar{K_1} - v_1\right)\left(\bar{K_2} - v_2\right) \ldots \left(\bar{K_n} - v_n\right)$$

Vector $\bar{K} = \left(\bar{K_1}, \bar{K_2}, \ldots, \bar{K_n}\right)$ is called Nash Bargaining solution.

Suppose, we use Nash Bargaining solution $\bar{K} = \left(\bar{K_1}, \bar{K_2}, \ldots, \bar{K_n}\right)$ in two stage game, this leads us to network $\bar{g}$, which is formed on the first stage, and subgame $\Gamma(\bar{g})$ on the second stage. Pair $(\bar{g}, \Gamma(\bar{g}))$ we shall call Nash Bargaining trajectory.

Proposition 1. *Nash Bargaining solution is time-consistent in G (two-stage game), if Nash Bargaining solution computed for game G coincides with Nash Bargaining solution computed for subgame $\Gamma(\bar{g})$. Nash Bargaining solution is time-consistent in game G.*

Proof. Consider the Nash Bargaining solution in two-stage game.Because players from the set $N \setminus i$ have the possibility not to form links with player i, the lower value of zero sum game v_i will be equal to 0, since player i can be isolated. $\bar{K} = \left(\bar{K_1}, \bar{K_2}, \ldots, \bar{K_n}\right)$ is the Nash Bargaining solution in two-stage game, and we have

$$\max_{K_1 \geq v_1, K_i \geq v_i, \ldots, K_n \geq v_n; (K_1, K_2, \ldots, K_n) \in K.} (K_1 - v_1) \ldots (K_n - v_n) = \\ = \left(\bar{K_1} - v_1\right)\left(\bar{K_2} - v_2\right) \ldots \left(\bar{K_n} - v_n\right) = \prod_{i=1}^{n} \bar{K_i},$$

Here K is the set of all possible payoffs in two stage game.

Consider the Nash Bargaining solution on the second stage.

$$\max_{K_1{}' \geq v_1{}', \ldots, K_n{}' \geq v_n{}'; (K_1{}', K_2{}', \ldots, K_n{}') \in K'.} (K_1{}' - v_1{}') \ldots (K_n{}' - v_n{}') = \\ = \left(\bar{K_1}' - v_1{}'\right)\left(\bar{K_2}' - v_2{}'\right) ... \left(\bar{K_n}' - v_n{}'\right) = \prod_{i=1}^{n} \bar{K_i}',$$

Where $K' \subset K$ is the set of all possible payoffs on the second stage.

Where $K' \subset K$ is the set of all possible payoffs on the second stage. The value of zero sum game $v_i{}'$ still will be equal to $\bar{v_i} = 0$ since player i can be isolated (because players can break links on the second stage). We have that $\bar{K} = \left(\bar{K_1}, \bar{K_2}, \ldots, \bar{K_n}\right)$ is the Nash Bargaining solution in two-stage game. Also along the Nash Bargaining trajectory of the cooperative game, $\bar{K}$ will remain in the subset K' $\left(\bar{K} \in K'\right)$. Hence, Nash Bargaining solution on the second stage $\bar{K}' = \left(\bar{K_1}', \bar{K_2}', \ldots, \bar{K_n}'\right)$ will always be equal to $\bar{K} = \left(\bar{K_1}, \bar{K_2}, \ldots, \bar{K_n}\right)$.

Therefore, Nash Bargaining solution is time-consistent in this model.

4. Example

In this section, we consider a three-person game as an illustration, i.e., the set of players $N = \{1, 2, 3\}$. Assume that player 1 can maintain 2 links and players 2, 3 can only maintain 1 link. Moreover player 3 can offer a link only to player 1. Under these restrictions, we have: subsets of players to whom each player can offer links are $M_1 = \{2,3\}, M_2 = \{1,3\}, M_3 = \{1\}$, a number of links each player can maintain: $a_1 = 2, a_2 = a_3 = 1$. Therefore, at the first stage sets of players' behaviors are: $G_1 = \{(0,0,0);(0,0,1);(0,1,0);(0,1,1)\}$, $G_2 = \{(0,0,0);(1,0,0);(0,0,1)\}$, $G_3 = \{(0,0,0);(1,0,0)\}$, and only four networks can be formed at the first stage of the game: the empty network (the network without links, $g = \emptyset$), $g = \{1,2\}$, $g = \{1,3\}$, $g = \{1,2,3\}$. Suppose that sets of controls U_i at the second stage for any network g realized at the first stage are the same $U_1 = U_2 = U_3 = \{A, B\}$, and payoff functions are defined as: $K_i(u_i)$: $K_i(A)$=0, $K_i(B)$=0; $i = 1,2,3$

$$K_1^{1,2}(u_1,u_2): \quad K_1^{1,2}(A,A) = 2; \quad K_1^{1,2}(A,B) = 4; \quad K_1^{1,2}(B,A) = 1; \quad K_1^{1,2}(B,B) = 3;$$

$$K_1^{1,3}(u_1,u_3): \quad K_1^{1,3}(A,A) = 3; \quad K_1^{1,3}(A,B) = 5; \quad K_1^{1,3}(B,A) = 1; \quad K_1^{1,3}(B,B) = 3;$$

$$K_2^{1,2}(u_2,u_1): \quad K_2^{1,2}(A,A) = 2; \quad K_2^{1,2}(A,B) = 4; \quad K_2^{1,2}(B,A) = 1; \quad K_2^{1,2}(B,B) = 3;$$

$$K_3^{1,3}(u_3,u_1): \quad K_3^{1,3}(A,A) = 2; \quad K_3^{1,3}(A,B) = 5; \quad K_3^{1,3}(B,A) = 1; \quad K_3^{1,3}(B,B) = 4;$$

$$K_1^{1,2;1,3}(u_1,u_2;u_1,u_3): \quad K_1^{1,2;1,3}(A,A,A) = 6; \quad K_1^{1,2;1,3}(A,A,B) = 7;$$
$$K_1^{1,2;1,3}(A,B,A) = 5; \quad K_1^{1,2;1,3}(A,B,B) = 2; \quad K_1^{1,2;1,3}(B,A,A) = 4;$$
$$K_1^{1,2;1,3}(B,A,B) = 6; \quad K_1^{1,2;1,3}(B,B,A) = 1; \quad K_1^{1,2;1,3}(B,B,B) = 9$$

Consider the network g=$\{1,2\}$:

$$\begin{pmatrix} (2,2,0) & (4,1,0) \\ (1,4,0) & (3,3,0) \end{pmatrix} \& \begin{pmatrix} (2,2,0) & (4,1,0) \\ (1,4,0) & (3,3,0) \end{pmatrix} \tag{4}$$

Here player 1 chooses the rows of the matrix (the first row corresponds to the choice of the strategy A and the second of B), player 2 choose the columns of the matrix (the first column corresponds to the choice of the strategy A and the second to B), and player 3 chooses one of the matrices (the first matrix corresponds to the choice of the strategy A and the second to B). In the described game,the Nash bargaining solution gives the payoffs $K_1(B,B) = K_2(B,B) = 3, K_3(A) = K_3(B) = 0$.

Strategy profiles are $(d_1^*(g), u_1^*) = ((0,1,0), B), (d_2^*(g), u_2^*) = ((1,0,0), B)$

Consider the network g=$\{1,3\}$:

$$\begin{pmatrix} (3,0,2) & (3,0,2) \\ (1,0,5) & (1,0,5) \end{pmatrix} \& \begin{pmatrix} (5,0,1) & (5,0,1) \\ (3,0,4) & (3,0,4) \end{pmatrix} \quad (5)$$

In the described game, the Nash bargaining solution gives the payoffs $K_1(B,B) = 3, K_3(B,B) = 4, K_2(A) = K_2(B) = 0$,

Strategy profiles are $(d_1^*(g), u_1^*) = ((0,0,1), B), (d_3^*(g), u_3^*) = ((1,0,0), B)$.

Consider the network g=$\{1,2,3\}$:

$$\begin{pmatrix} (6,2,2) & (5,2,2) \\ (4,4,5) & (1,3,5) \end{pmatrix} \& \begin{pmatrix} (7,2,1) & (2,1,1) \\ (6,4,4) & (9,3,4) \end{pmatrix} \quad (6)$$

In the described game, the Nash bargaining solution gives the payoffs $K_1(B,B,B) = 9, K_2(B,B,B) = 3, K_3(B,B,B) = 4$,

Strategy profiles are $(d_1^*(g), u_1^*) = ((0,1,1), B), (d_2^*(g), u_2^*) = ((1,0,0), B)$ $(d_3^*(g), u_3^*) = ((0,0,1), B)$.

Now, consider two stage game, the Nash bargaining solution gives the payoffs $K_1(B,B,B) = 9, K_2(B,B,B) = 3, K_3(B,B,B) = 4$, and strategy profiles are $(g_1^*, d_1^*(g), u_1^*) = ((0,1,1), (0,1,1), B), (g_2^*, d_2^*(g), u_2^*) = ((1,0,0), (1,0,0), B)$, $(g_3^*, d_3^*(g), u_3^*) = ((0,0,1), (0,0,1), B)$. In the subgame starting from the second stage, after realized Nash Bargaining solution computed for two stage game on the first stage. we obtain the Nash bargaining solution which is equal $(9,3,4)$. Moreover, Nash bargaining solution coincide with $(9,3,4)$ which is the Nsah bargaining solution in the game starting from the first stage. We see that the Nash bargaining solution is a time-consistent solution concept.

5. The classical two-stage network game

The first stage, is as in previous case. However, at the second stage, we do not give the players opportunity to revise the network. So they just choose control at the second stage. Player $i \in N$ chooses only controls u_i form a finite set U_i.

A payoff function K_i of player $i \in N$ depends on both network g and controls $u_i, i \in N$. Specifically, it depends on player i's behavior at the second stage as well as behavior of his neighbors in the network g. i.e., $K_i\left(u_i, u_{N_i(g)}\right)$ is a nonnegative real-valued function defined on $U_i \times \prod_{j \in N_i(g)} U_j$. Here u_{N_i} denotes a profile of controls u_j chosen by all neighbors $j \in N_i(g)$ of players i in the network g.

Proposition 2. *Nash Bargaining solution is time-inconsistent in G (two-stage game)*

Proof. Consider the Nash Bargaining solution in two-stage game.

Because players from the set $N \setminus i$ have the possibility not to form links with player i. The lower value of zero sum game v_i will be equal to 0, since player i can be isolated. $\bar{K} = \left(\bar{K_1}, \bar{K_2}, \ldots, \bar{K_n}\right)$ is the Nash Bargaining solution in two-stage game, and we have

$$\max_{K_1, K_i, \ldots, K_n; K_1 \geq v_1, K_i \geq v_i, \ldots, K_n \geq v_n; (K_1, K_2, \ldots, K_n) \in K.} (K_1 - v_1) \ldots (K_n - v_n) =$$

$$= \left(\bar{K_1} - v_1\right)\left(\bar{K_2} - v_2\right) \ldots \left(\bar{K_n} - v_n\right) = \prod_{i=1}^{n} \bar{K_i},$$

Here K is the set of all possible payoffs in two stage game.
Consider the Nash Bargaining solution on the second stage.

$$\max_{K_1{}',K_i{}',\dots,K_n{}';K_1{}'\geq v_1{}',K_i{}'\geq v_i{}',\dots,K_n{}'\geq v_n{}';(K_1{}',K_2{}',\dots,K_n{}')\in K'.} (K_1{}'-v_1{}')\cdot$$

$$\cdot(K_i{}'-v_i{}')\dots(K_n{}'-v_n{}')=\left(\bar{K_1}{}'-v_1{}'\right)\left(\bar{K_2}{}'-v_2{}'\right)\dots\left(\bar{K_n}{}'-v_n{}'\right)=\prod_{i=1}^{n}\bar{K_i}{}',$$

Where $K'\subset K$ is the set of all possible payoffs on the second stage. The value of zero sum game $v_i{}'$ will be $v_i'=\max_i\min_{N-i}K_i'(u_i;u_{\{N_i(g^d)\}})$, We have that $\bar{K}=\left(\bar{K_1},\bar{K_2},\dots,\bar{K_n}\right)$ is the Nash Bargaining solution in two-stage game where v_i will be equal to 0. Also along the Nash Bargaining trajectory of the cooperative game, $\bar{K}$ will remain in the subset $K'\left(\bar{K}\in K'\right)$. So, obviously, $\bar{K}=\left(\bar{K_1},\bar{K_2},\dots,\bar{K_n}\right)$ is not coincides with $\left(\bar{K_1}{}',\bar{K_2}{}',\dots,\bar{K_n}{}'\right)$ Therefore, Nash Bargaining solution is time-inconsistent in this model.

5.1. Example

By using the same example, we can get the following: Consider the network g=$\{1,2\}$:

$$\begin{pmatrix}(2,2,0) & (4,1,0)\\(1,4,0) & (3,3,0)\end{pmatrix}\ \&\ \begin{pmatrix}(2,2,0) & (4,1,0)\\(1,4,0) & (3,3,0)\end{pmatrix} \tag{7}$$

Here player 1 chooses the rows of the matrix (the first row corresponds to the choice of the strategy A and the second of B), player 2 choose the columns of the matrix (the first column corresponds to the choice of the strategy A and the second of B), and player 3 chooses one of the matrices (the first matrix corresponds to the choice of the strategy A and the second of B). In the described game,the Nash bargaining solution gives the payoffs $K_1(B,B)=K_2(B,B)=3,K_3(A)=K_3(B)=0$.

Consider the network g=$\{1,3\}$:

$$\begin{pmatrix}(3,0,2) & (3,0,2)\\(1,0,5) & (1,0,5)\end{pmatrix}\ \&\ \begin{pmatrix}(5,0,1) & (5,0,1)\\(3,0,4) & (3,0,4)\end{pmatrix} \tag{8}$$

In the described game, the Nash bargaining solution gives the payoffs $K_1(B,B)=3,K_3(B,B)=4,K_2(A)=K_2(B)=0$.
Consider the network g=$\{1,2,3\}$:

$$\begin{pmatrix}(6,2,2) & (5,2,2)\\(4,4,5) & (1,3,5)\end{pmatrix}\ \&\ \begin{pmatrix}(7,2,1) & (2,1,1)\\(6,4,4) & (9,3,4)\end{pmatrix} \tag{9}$$

In the described game, the Nash bargaining solution gives the payoffs $K_1(B,A,B)=6,K_2(B,A,B)=4,K_3(B,A,B)=4$.

Now, consider two stage game, the Nash bargaining solution gives the payoffs $K_1(B,B,B)=9,K_2(B,B,B)=3,K_3(B,B,B)=4$.

In the subgame starting from the second stage, after realized Nash Bargaining solution computed for two stage game on the first stage. we obtain the Nash bargaining solution which is equal $(6,4,4)$. Therefore, Nash bargaining solution is not coincide with $(9,3,4)$ which is the Nsah bargaining solution in the two stage game. We see that the Nash bargaining solution is not a time-consistent solution concept.

6. Conclusion

In the special class of two-stage network where players have the opportunity to revise their network which they formed before, Nash Bargaining solution is time-consistent. We also consider in the classical two-stage network game where players are just choosing controls at the second stage. Cooperative setting is considered. In the cooperative case, we use Nash Bargaining Solution as a solution concept. It is demonstrated that the Nash Bargaining Solution satisfies the time consistency property for this special class of two-stage network game. But it does not satisfy the time consistency property for the classical two-stage network game.

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