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On the Characteristic Function Construction Technique in Differential Games with Prescribed and Random Duration*

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Abstract This paper focuses on different approaches for calculating characteristic functions in cooperative differential games with prescribed and random duration. We construct $\alpha-$, $\delta-$ and $\zeta-$characteristic functions and examine their properties in the differential game of pollution control. Additionally, we introduce a new $\eta-$ characteristic function.

Keywords: differential games, prescribed duration, random duration, characteristic function, environmental resource management, pollution control.

1. Introduction

In cooperative differential games the problem of the calculation of a characteristic function (c.f.) plays an important role since the characteristic function allows to obtain optimal (cooperative) solution of the game. The review and analysis of different methods for construction of c.f. for cooperative games with so-called negative externalities (see, e.g., Chander, 2007) in static formulation was presented in (Reddy and Zaccour, 2016). Whereas in a dynamic formulation the construction of α-, δ- characteristic functions (see, correspondingly, Petrosjan and Danilov, 1982, Petrosjan and Zaccour, 2003) was analyzed in (Gromova and Petrosyan, 2017) where a new approach to constructing ζ-characteristic function was introduced (see also (Petrosyan and Gromova, 2014) for the first reference in Russian). This paper presents different techniques for characteristic function construction in cooperative differential game with prescribed (Petrosjan and Danilov, 1982) and random duration as well. To illustrate our results we consider the dynamic game-theoretical problem of pollution control (Gromova, 2016) which belongs to the class of games with negative externalities.

2. Different techniques of the characteristic function construction

Let $K_i(\cdot)$ and $u_i \in U_i \subseteq compR^l$ be the payoff functions and controls in a classical cooperative differential game of n players with prescribed duration (Petrosjan and Danilov, 1982). Assume that all standard restrictions (Krasovskii and Subbotin, 1988) on the parameters, controls and trajectory function are satisfied. To define the cooperative game we have to construct the characteristic function $V(S, \cdot)$ for

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every coalition $S \subseteq N$ in the game. The c.f. in a cooperative game is a mapping from the set of all possible coalitions:

$$V(\cdot) : 2^N \to R, \qquad V(\emptyset) = 0.$$

Note that the value of the c.f. for the grand coalition N equals to $V(N, \cdot)$.

The value $V(S)$ is interpreted traditionally as the power of the coalition S. The important property of a c.f. is *superadditivity*:

$$V(S_1 \cup S_2) \geq V(S_1) + V(S_2), \qquad \forall S_1, S_2 \subseteq N,\ S_1 \cap S_2 = \emptyset. \tag{1}$$

The use of superadditive characteristic functions in solving various problems in the field of cooperative game theory in static and dynamic setting provides a number of advantages (Gromova and Petrosyan, 2017).

There are several approaches to the construction of characteristic functions (see Neumann and Morgenstern, 1953, Reddy and Zaccour, 2016, Petrosjan and Zaccour, 2003, Gromova and Petrosyan, 2017). In this paper we shall focus on the α-, δ-, and ζ-characteristic functions and then introduce the new one named $\eta-$characteristic function.

2.1. α-characteristic function

A classical way to define the c.f. is to use the lower value of the zero-sum game between the coalition S, acting as the first (maximizing) player and the coalition $N \setminus S$, acting as the second (minimizing) player. This approach had been introduced in (Neumann and Morgenstern, 1953) and now is called α-characteristic function.

We have

$$V^{\alpha}(S, \cdot) = \begin{cases} 0, & S = \{\emptyset\}, \\ \max\limits_{\substack{u_i, \\ i \in S}} \min\limits_{\substack{u_j, \\ j \in N \setminus S}} \sum\limits_{i \in S} K_i(u_1, \ldots, u_n, \cdot), & S \subset N, \\ \max\limits_{u_1, \ldots u_n} \sum\limits_{i=1}^{n} K_i(u_1, \ldots, u_n, \cdot), & S = N, \end{cases} \tag{2}$$

where $\max\limits_{\substack{u_i, \\ i \in S}} \min\limits_{\substack{u_j, \\ j \in N \setminus S}}$ is the lower value of the zero-sum game between the coalition S and $N \setminus S$ with the payoff function $\sum\limits_{i \in S} K_i(u_1, \ldots, u_n, \cdot)$. It was proved in (Petrosjan and Danilov, 1982) that in general the α-characteristic function is superadditive.

2.2. δ-characteristic function

δ-characteristic function, (Petrosjan and Zaccour, 2003), of a coalition S is constructed in two steps. First, we calculate the Nash equilibrium strategies for all players. Second, we fix (freeze) those strategies for players from $N \setminus S$ while the players from S seek to maximize their joint payoff $\sum\limits_{i \in S} K_i(u_1, \ldots, u_n, \cdot)$.

$$V^{\delta}(S, \cdot) = \begin{cases} 0, & S = \{\emptyset\}, \\ \max\limits_{\substack{u_i,\ i \in S \\ u_j = u_j^{NE},\ j \in N \setminus S}} \sum\limits_{i \in S} K_i(u_S, u_{N \setminus S}^{NE}, \cdot), & S \subset N, \\ \max\limits_{u_1, \ldots u_n} \sum\limits_{i=1}^{n} K_i(u_1, \ldots, u_n, \cdot), & S = N. \end{cases} \tag{3}$$

In general, a δ-characteristic function is a non-superadditive function (see examples in (Gromova et al., 2017)).

2.3. ζ-characteristic function

One of the novel approaches is to use a ζ-characteristic function, (Gromova and Petrosyan, 2017). This c.f. of coalition S is computed in two stages: first, we find optimal controls maximizing the total payoff of the players; next, the cooperative optimal strategies are used by the players from the coalition S while the left-out players from $N \setminus S$ use the strategies minimizing the total payoff of the players from S. We have

$$V^{\zeta}(S,\cdot) = \begin{cases} 0, & S = \{\emptyset\}, \\ \min\limits_{\substack{u_j \in U_j,\, j \in N \setminus S, \\ u_i = u_i^*,\, i \in S}} \sum\limits_{i \in S} K_i(u_S^*, u_{N\setminus S}, \cdot), & S \subset N, \\ \max\limits_{\substack{u_1,\ldots,u_n, \\ u_i \in U_i,\, i \in N}} \sum\limits_{i=1}^{n} K_i(u_1,\ldots,u_n,\cdot), & S = N, \end{cases} \tag{4}$$

where $u^* = \{u_i^*\}_{i \in N}$ is the profile of strategies for which the maximal value of payoff function is achieved for all players, $u_S^* = \{u_i^*\}_{i \in S}$.

The c.f. defined in this way is in general superadditive (see Gromova and Petrosyan, 2017). Note, that for the case of ζ-characteristic function players from $N \setminus S$ have active reaction while players from S just use the same strategies u_i^*, $i \in S$ as they were use in the case of total payoff maximization.

3. Game-theoretical model of pollution control with prescribed duration

Consider a game-theoretic model of pollution control based on the models published in (Breton et al., 2005), see also (Gromova, 2016). There are 3 players (companies, countries) that participate in the game, $N = \{1, 2, 3\}$. Each player has an industrial production site. It is assumed that the production is proportional to the pollution u_i. Thus, the strategy of a player is to choose the amount of pollutions emitted to the atmosphere, $u_i \in [0; b_i]$. In this example the solution will be considered in the class of open-loop strategies $u_i(t)$ and Pontryagin's maximum principle (PMP) is applied (Pontryagin, 2018).

The dynamics of the total amount of pollution $x(t)$ is described by following equation

$$\dot{x}(t) = \sum_{i=1}^{3} u_i(t).$$

The instantaneous payoff of i-th player is defined as:

$$R(u_i(t)) = b_i u_i(t) - \frac{1}{2} u_i^2(t), \qquad i \in N.$$

Each player has to bear expenses due to the pollution removal. Hence the instantaneous payoff (utility) of the i-th player is equal to $R(u_i(t)) - d_i x(t), d_i \geq 0$. The payoff of the i-th player is thus defined as

$$K_i(x_0, T - t_0, u_1, u_2, u_3) = \int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2} u_i \Big) u_i - d_i x \Big) dt.$$

We will assume additional regularity constraints to be satisfied: $\forall i \in N$ $b_i \geq D_N(T-t_0)$, where $D_N = \sum\limits_{i=1}^{3} d_i$.

3.1. Construction of characteristic functions

To find the profile of optimal strategies we have to solve the maximization problem

$$\max_{u_1,u_2,u_3} \sum_{i=1}^{3} \int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i - d_i x\Big)dt.$$

The Hamiltonian is

$$H(x,u,\psi) = \sum_{i=1}^{3} \Big(b_i - \frac{1}{2}u_i\Big)u_i - \sum_{i=1}^{3} d_i x + \psi \sum_{i=1}^{3} u_i,$$

its first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H}{\partial u_i}(x,u,\psi) = b_i - u_i + \psi, \quad i = \overline{1,3}.$$

The Hessian matrix is negative definite hence we conclude that Hamiltonian H is concave w.r.t. u_i.

$$\frac{\partial^2 H}{\partial u_i^2}(x,u,\psi) = -1 < 0, \quad i = \overline{1,3}.$$

The adjoint equations and the related transversality conditions are

$$\begin{cases} \frac{d\psi}{dt} = \sum\limits_{i=1}^{3} d_i, \\ \psi(T) = 0. \end{cases}$$

We get the optimal control

$$u^*(t) = (b_1 - D_N(T-t), \quad b_2 - D_N(T-t), \quad b_3 - D_N(T-t)). \tag{5}$$

Given the initial conditions (t,x) the value of c.f. for the grand coalition N can be written as

$$\begin{aligned} &V^{\alpha}(N,x(t),T-t) = V^{\delta}(N,x(t),T-t) = V^{\zeta}(N,x(t),T-t) = \\ &= -D_N(T-t)x(t) + \tfrac{1}{2}\tilde{B}_N(T-t) - \tfrac{1}{2}D_N B_N(T-t)^2 + \tfrac{1}{2}D_N^2(T-t)^3, \end{aligned} \tag{6}$$

where $B_N = \sum\limits_{i=1}^{3} b_i$, $\tilde{B}_N = \sum\limits_{i=1}^{3} b_i^2$.

Now we have to calculate the value of the characteristic function for coalitions of one and two players. Let us carry out some preliminary calculations.

For the case of $\delta-$c.f. we have to find the Nash equilibrium. To find Nash equilibrium strategies for this game one has to maximize the payoff for each player i by the control u_i in assumption that another players use fixed NE strategies

$$\max_{u_i} \int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i - d_i x\Big)dt, \quad i = 1,2,3.$$

We apply the PMP to this maximization problem. The players' Hamiltonians are

$$H_i(x,u,\psi) = \Big(b_i - \frac{1}{2}u_i\Big)u_i - d_i x + \psi \sum_{i=1}^{3} u_i, \quad i = \overline{1,3},$$

and their first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H_i}{\partial u_i}(x,u,\psi) = b_i - u_i + \psi, \quad i = \overline{1,3}.$$

The Hessian matrices are negative definite hence we conclude that Hamiltonians H_i are concave w.r.t. u_i.

$$\frac{\partial^2 H_i}{\partial u_i^2}(x,u,\psi) = -1 < 0, \quad i = \overline{1,3}.$$

The adjoint equations and the related transversality conditions are

$$\begin{cases} \frac{d\psi}{dt} = d_i, \\ \psi(T) = 0. \end{cases}$$

We get the Nash equilibrium strategies

$$u^{NE}(t) = (b_1 - d_1(T-t), \quad b_2 - d_2(T-t), \quad b_3 - d_3(T-t)). \tag{7}$$

According (3) we have to solve the maximization problem, taking into account (7). We have:

$$\max_{\substack{u_i,u_j \\ u_k = u_k^{NE}}} \int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i + (b_j - \frac{1}{2}u_j\Big)u_j - (d_i + d_j)x\Big)dt.$$

Following PMP we get

$$u_i^S = b_i - (d_i + dj)(T-t), \quad u_j^S = b_j - (d_i + dj)(T-t). \tag{8}$$

For the case of $\alpha-$, $\zeta-$ c.f. (2), (4) we will have to solve the minimization problem

$$\min_{u_k} \int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i + (b_j - \frac{1}{2}u_j\Big)u_j - (d_i + d_j)x\Big)dt.$$

By using PMP we get

$$u_k = b_k. \tag{9}$$

According to the definition of the α-c.f. (2), to construct it we have to solve the maximization problem with (9):

$$\max_{\substack{u_i,u_j \\ u_k = b_k}} \int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i + (b_j - \frac{1}{2}u_j\Big)u_j - (d_i + d_j)x\Big)dt.$$

Applying PMP in the same way we get

$$u_i^S = b_i - (d_i + dj)(T-t), \quad u_j^S = b_j - (d_i + dj)(T-t). \tag{10}$$

Given the initial conditions (t, x) one can calculate the value of three above mentioned characteristic functions for coalitions of one and two players.

Considering (2), (9), (10) we obtain the value of a $\alpha-$characteristic function:

$$V^{\alpha}(\{i\}, x(t), T-t) = -d_i(T-t)x(t) + \frac{1}{2}b_i^2(T-t) - \frac{1}{2}B_N d_i(T-t)^2 + \frac{1}{6}d_i^2(T-t)^3, \quad (11)$$

$$\begin{aligned} V^{\alpha}(\{i,j\}, x(t), T-t) = -(d_i+d_j)(T-t)x(t) + \tfrac{1}{2}(b_i^2+b_j^2)(T-t)- \\ -\tfrac{1}{2}B_N(d_i+d_j)(T-t)^2 + \tfrac{1}{3}(d_i+d_j)^2(T-t)^3. \end{aligned} \quad (12)$$

We get a $\delta-$characteristic function from (3), (7), (8):

$$\begin{aligned} V^{\delta}(\{i\}, x(t), T-t) = -d_i(T-t)x(t) + \tfrac{1}{2}b_i^2(T-t) - \tfrac{1}{2}B_N d_i(T-t)^2+ \\ +\tfrac{1}{6}d_i(2D_N - d_i)(T-t)^3, \end{aligned} \quad (13)$$

$$\begin{aligned} V^{\delta}(\{i,j\}, x(t), T-t) = -(d_i+d_j)(T-t)x(t) + \tfrac{1}{2}(b_i^2+b_j^2)(T-t)- \\ -\tfrac{1}{2}B_N(d_i+d_j)(T-t)^2 + \tfrac{1}{3}(d_k(d_i+d_j)+(d_i+d_j)^2)(T-t)^3. \end{aligned} \quad (14)$$

Finally, for a $\zeta-$characteristic function from (4), (5), (9) we obtain:

$$\begin{aligned} V^{\zeta}(\{i\}, x(t), T-t) = -d_i(T-t)x(t) + \tfrac{1}{2}b_i^2(T-t) - \tfrac{1}{2}B_N d_i(T-t)^2- \\ -\tfrac{1}{6}D_N(D_N - 2d_i)(T-t)^3, \end{aligned} \quad (15)$$

$$\begin{aligned} V^{\zeta}(\{i,j\}, x(t), T-t) = -(d_i+d_j)(T-t)x(t) + \tfrac{1}{2}(b_i^2+b_j^2)(T-t)- \\ -\tfrac{1}{2}B_N(d_i+d_j)(T-t)^2 - \tfrac{1}{3}D_N(D_N - 2(d_i+d_j))(T-t)^3. \end{aligned} \quad (16)$$

3.2. Superadditivity of the c.f.

Check that $\alpha-$characteristic function (2) is superadditive (1). From (6), (11), (12) we have

$$V^{\alpha}(N) - V^{\alpha}(\{k\}) - V^{\alpha}(\{i,j\}) = \frac{2}{3}d_k(d_i+d_j)(T-t)^3 \geq 0,$$

$$V^{\alpha}(\{i,j\}) - V^{\alpha}(\{i\}) - V^{\alpha}(\{j\}) = \frac{2}{3}d_i d_j(T-t)^3 \geq 0,$$

since $t \in [t_0, T]$. We conclude that the constructed function is superadditive which agrees with the previously obtained results.

At the next step we check if $\delta-$characteristic function (3) is superadditive. From (6), (13), (14) we have

$$V^{\delta}(N) - V^{\delta}(\{k\}) - V^{\delta}(\{i,j\}) = \frac{1}{6}(D_N^2 + d_k^2)(T-t)^3 \geq 0,$$

$$V^\delta(\{i,j\}) - V^\delta(\{i\}) - V^\delta(\{j\}) = \frac{1}{6}(d_i^2 + d_j^2)(T-t)^3 \geq 0,$$

since $t \in [t_0, T]$. This proves the superadditivity of the constructed $\delta-$ c.f.

Finally we may check if $\zeta-$characteristic function is also superadditive in this game. From (6), (15), (16) we have

$$V^\zeta(N) - V^\zeta(\{k\}) - V^\zeta(\{i,j\}) = \frac{1}{3}D_N(d_i + d_j + 2d_k)(T-t)^3 \geq 0,$$

$$V^\zeta(\{i,j\}) - V^\zeta(\{i\}) - V^\zeta(\{j\}) = \frac{1}{3}D_N(d_i + d_j)(T-t)^3 \geq 0,$$

since $t \in [t_0, T]$. This concludes that $\zeta-$characteristic function is also superadditive (as must be in general).

3.3. Comparison of characteristic functions

Let us investigate how the constructed c.f.s relate to each other. Obviously, we have

$$V^\delta(N) = V^\alpha(N),$$

also from (11), (13) and (12), (14) we get

$$V^\delta(\{i\}) = V^\alpha(\{i\}) + \frac{1}{3}d_i(d_j + d_k)(T-t)^3,$$

$$V^\delta(\{i,j\}) = V^\alpha(\{i,j\}) + \frac{1}{3}d_k(d_i + d_j)(T-t)^3.$$

Thus,

$$V^\delta(\cdot) \geq V^\alpha(\cdot). \tag{17}$$

Furthermore, from (11), (15) and (12), (16) we have

$$V^\alpha(N) = V^\zeta(N),$$

$$V^\alpha(\{i\}) = V^\zeta(\{i\}) + \frac{1}{6}(d_j + d_k)^2(T-t)^3,$$

$$V^\alpha(\{i,j\}) = V^\zeta(\{i,j\}) + \frac{1}{3}d_k^2(T-t)^3.$$

Thus,

$$V^\alpha(\cdot) \geq V^\zeta(\cdot). \tag{18}$$

Finally, (17) and (18) imply that

$$V^\delta(\cdot) \geq V^\alpha(\cdot) \geq V^\zeta(\cdot).$$

4. Game-theoretical model of pollution control with random duration

To make the model from Sec. 3 more realistic we examine the game-theoretic model of pollution control with random duration (Petrosyan and Murzov, 1966, Petrosyan and Shevkoplyas, 2000, Petrosyan and Shevkoplyas, 2003, Marin-Solano and Shevkoplyas, 2011, Shevkoplyas, 2014) $(T - t_0)$, where T is a random variable with exponential distribution function $F(t), t \in [t_0, T_f]$. The strategy of each player is to choose the amount of pollution emitted to the atmosphere, $u_i \in [0; b_i]$. Let us consider case of $N = \{1, 2, 3\}$. The game starts from initial state x_0 at the time t_0.

The dynamics of the total amount of pollution $x(t)$ is described by

$$\dot{x}(t) = \sum_{i=1}^{3} u_i(t), \quad x(t_0) = x_0.$$

The expectation of the payoff of players $i = 1, 2, 3$ are calculated as

$$K_i(x_0, t_0, T_f, u_1, u_2, u_3) = E\Big(\int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i(\tau)\Big) u_i(\tau) - d_i x(\tau) \Big) d\tau \Big).$$

We assume that $d_i \geq 0$, $\forall i = 1, 2, 3$ and there are additional constraints called the regularity constraints: $b_i \geq \frac{D_N}{\lambda}$, where $D_N = \sum\limits_{i=1}^{3} d_i$.

According to (Shevkoplyas, 2014, Kostyunin and Shevkoplyas, 2011, Gromova and Tur, 2017) the payoff of each player $i \in N$ can be represented as

$$K_i(x_0, t_0, T_f, u_1, u_2, u_3) = \int_{t_0}^{\infty} \Big(\Big(b_i - \frac{1}{2}u_i(\tau)\Big) u_i(\tau) - d_i x(\tau) \Big) e^{-\lambda(\tau - t_0)} d\tau =$$

$$= e^{\lambda t_0} \int_{t_0}^{\infty} \Big(\Big(b_i - \frac{1}{2}u_i(\tau)\Big) u_i(\tau) - d_i x(\tau) \Big) e^{-\lambda \tau} d\tau.$$

We consider this game in cooperative form and construct the c.f. by three described above methods.

4.1. Construction of characteristic functions

We start with solving the maximization problem in order to find the profile of optimal strategies.

$$\max_{u_1, u_2, u_3} \sum_{i=1}^{3} e^{\lambda t_0} \int_{t_0}^{\infty} \Big(\Big(b_i - \frac{1}{2}u_i\Big) u_i - d_i x \Big) e^{-\lambda t} dt.$$

The Hamiltonian is

$$H(x, u, \psi) = \sum_{i=1}^{3} \Big(\Big(b_i - \frac{1}{2}u_i\Big) u_i - d_i x \Big) e^{-\lambda t} + \psi \sum_{i=1}^{3} u_i,$$

its first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H}{\partial u_i}(x, u, \psi) = (b_i - u_i) e^{-\lambda t} + \psi, \quad i = \overline{1, 3}.$$

The Hessian matrix is negative definite hence we conclude thet Hamiltonian H is concave w.r.t. u_i.

$$\frac{\partial^2 H}{\partial u_i^2}(x,u,\psi) = -e^{-\lambda t} < 0, \quad i=\overline{1,3}.$$

The adjoint equations and the related transversality conditions are

$$\begin{cases} \frac{d\psi}{dt} = \sum\limits_{i=1}^{3} d_i e^{-\lambda t}, \\ \lim\limits_{t\to\infty} \psi(t) = 0. \end{cases}$$

We have optimal controls in the following form

$$u^*(t) = \Big(b_1 - \frac{D_N}{\lambda}, b_2 - \frac{D_N}{\lambda}, b_3 - \frac{D_N}{\lambda}\Big). \tag{19}$$

Given the initial conditions (t,x), we compute the c.f. for the grand coalition N from (19).

$$\begin{aligned} &V^\alpha(N,x(t),T-t) = V^\delta(N,x(t),T-t) = V^\zeta(N,x(t),T-t) = \\ &= \tfrac{1}{\lambda^3}\Big(\tfrac{\tilde{B}_N}{2}\lambda^2 - B_N D_N \lambda + 1\tfrac{1}{2}D_N^2 - D_N\lambda^2 x\Big), \end{aligned} \tag{20}$$

where $B_N = \sum\limits_{i=1}^{3} b_i$, $\tilde{B}_N = \sum\limits_{i=1}^{3} b_i^2$.

Next, we construct the characteristic function for coalitions of one and two players. First, we find the Nash equilibrium to construct $\delta-$c.f.

$$\max_{u_i} \int_{t_0}^{\infty} \Big(\Big(b_i - \frac{1}{2}u_i(\tau)\Big)u_i(\tau) - d_i x(\tau)\Big)e^{-\lambda\tau}d\tau.$$

According to the PMP the Hamiltonian has the form

$$H_i(x,u,\psi) = \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i - d_i x\Big)e^{-\lambda t} + \psi\sum_{i=1}^{3} u_i, \quad i=\overline{1,3}$$

and its first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H_i}{\partial u_i}(x,u,\psi) = (b_i - u_i)e^{-\lambda t} + \psi, \quad i=\overline{1,3}.$$

The Hamiltonians H_i are concave w.r.t. u_i, because the respective Hessian matrices are negative definite.

$$\frac{\partial^2 H_i}{\partial u_i^2}(x,u,\psi) = -e^{-\lambda t} < 0, \quad i=\overline{1,3}.$$

The adjoint equations and the related transversality conditions are

$$\begin{cases} \frac{d\psi}{dt} = d_i e^{-\lambda t}, \\ \lim\limits_{t\to\infty} \psi(t) = 0. \end{cases}$$

We get the Nash equilibrium strategies

$$u^{NE}(t) = \Big(b_1 - \frac{d_1}{\lambda}, \quad b_2 - \frac{d_2}{\lambda}, \quad b_3 - \frac{d_3}{\lambda}\Big). \tag{21}$$

Now we solve the maximization problem while considering (21).

$$\max_{\substack{u_i,u_j\\ u_k=u_k^{NE}}} \int_{t_0}^{T} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i + (b_j - \frac{1}{2}u_j\Big)u_j - (d_i+d_j)x\Big)dt.$$

Following PMP we get the controls in form of

$$u_i^S = b_i - \frac{d_i+d_j}{\lambda}, \quad u_j^S = b_j - \frac{d_i+d_j}{\lambda}. \tag{22}$$

For the case of $\alpha-$, $\zeta-$c.f. (2), (4) we will have to solve the minimization problem

$$\min_{u_k} e^{\lambda t_0} \int_{t_0}^{\infty} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i + \Big(b_j - \frac{1}{2}u_j\Big)\Big)u_j - (d_i+d_j)x\Big)e^{-\lambda t}dt.$$

From the PMP we get

$$u_k = b_k. \tag{23}$$

According to the definition of α-c.f. (2), to calculate it we have to solve the maximization problem having in mind (23):

$$\max_{\substack{u_i,u_j\\ u_k=b_k}} \int_{t_0}^{\infty} \Big(\Big(b_i - \frac{1}{2}u_i\Big)u_i + (b_j - \frac{1}{2}u_j\Big)u_j - (d_i+d_j)x\Big)e^{-\lambda t}dt.$$

Applying the PMP to this problem we get

$$u_i^S = b_i - \frac{d_i+d_j}{\lambda}, \quad u_j^S = b_j - \frac{d_i+d_j}{\lambda}. \tag{24}$$

Given the initial conditions (t, x), we calculate the value of three described above characteristic functions for one- and two-players coalitions.

With (2), (23), (24) we calculate the $\alpha-$c.f.

$$V^{\alpha}(\{i\}, x(t), T-t) = \frac{1}{\lambda^3}\Big(\frac{1}{2}b_i^2\lambda^2 - B_N d_i\lambda + \frac{1}{2}d_i^2 - d_i\lambda^2 x(t)\Big), \tag{25}$$

$$\begin{aligned} V^{\alpha}(\{i,j\}, x(t), T-t) = \tfrac{1}{\lambda^3}\Big(\tfrac{1}{2}(b_i^2+b_j^2)\lambda^2 - B_N(d_i+d_j)\lambda + (d_i+d_j)^2 - \\ -(d_i+d_j)\lambda^2 x(t)\Big). \end{aligned} \tag{26}$$

According (3), (21), (22) the $\delta-$characteristic function could be calculated as

$$V^{\delta}(\{i\}, x(t), T-t) = \frac{1}{\lambda^3}\Big(\frac{1}{2}b_i^2\lambda^2 - B_N d_i\lambda + (d_j+d_k)d_i + \frac{1}{2}d_i^2 - d_i\lambda^2 x(t)\Big), \tag{27}$$

$$V^\delta(\{i,j\},x(t),T-t)=\tfrac{1}{\lambda^3}\Big(\tfrac{1}{2}(b_i^2+b_j^2)\lambda^2-B_N(d_i+d_j)\lambda+d_k(d_i+d_j)+ \\ +(d_i+d_j)^2-(d_i+d_j)\lambda^2x(t)\Big). \tag{28}$$

For the $\zeta-$characteristic function from (4), (19), (23) we obtain

$$V^\zeta(\{i\},x(t),T-t)=\frac{1}{\lambda^3}\Big(\frac{1}{2}b_i^2\lambda^2-B_Nd_i\lambda-\frac{1}{2}D_N^2+d_iD_N-d_i\lambda^2x(t)\Big), \tag{29}$$

$$V^\zeta(\{i,j\},x(t),T-t)=\tfrac{1}{\lambda^3}\Big(\tfrac{1}{2}(b_i^2+b_j^2)\lambda^2-B_N(d_i+d_j)\lambda- \\ -D_N^2+2D_N(d_i+d_j)-(d_i+d_j)\lambda^2x(t)\Big). \tag{30}$$

4.2. Superadditivity of c.f.

It is easy to check the superadditivity of $\alpha-$characteristic function by (20), (25), (26).

$$V^\alpha(N)-V^\alpha(\{k\})-V^\alpha(\{i,j\})=\frac{1}{2\lambda^3}\Big((d_i+d_j)^2+2d_k^2+6d_k(d_i+d_j)\Big)\geq 0,$$

$$V^\alpha(\{i,j\})-V^\alpha(\{i\})-V^\alpha(\{j\})=\frac{1}{2\lambda^3}\Big(d_i^2+d_j^2+4d_id_j\Big)\geq 0,$$

since $t\in[t_0,T]$.

Additionally we could prove that $\delta-$characteristic function is superadditive using (20), (27), (28).

$$V^\delta(N)-V^\delta(\{k\})-V^\delta(\{i,j\})=\frac{1}{2\lambda^3}\Big((d_i+d_J)^2+2d_k^2+2d_k(d_i+d_j)\Big)\geq 0,$$

$$V^\delta(\{i,j\})-V^\delta(\{i\})-V^\delta(\{j\})=\frac{1}{2\lambda^3}\Big(d_i^2+d_j^2\Big)\geq 0,$$

since $t\in[t_0,T]$,

Also $\zeta-$characteristic function is superadditive too (20), (29), (30).

$$V^\zeta(N)-V^\zeta(\{k\})-V^\zeta(\{i,j\})=\frac{1}{\lambda^3}D_N\Big(d_i+d_j+2d_k\Big)\geq 0,$$

$$V^\zeta(\{i,j\})-V^\zeta(\{i\})-V^\zeta(\{j\})=\frac{1}{\lambda^3}D_N\Big(d_i+d_j\Big)\geq 0,$$

since $t\in[t_0,T]$.

4.3. Comparison of characteristic functions

Next we are going to examine the properties of the constructed c.f. with respect to each other. It is clear that

$$V^{\delta}(N) = V^{\alpha}(N),$$

also from (25), (27) and (26), (28) we get

$$V^{\delta}(\{i\}) = V^{\alpha}(\{i\}) + \frac{(d_j + d_k)d_i}{\lambda^3},$$

$$V^{\delta}(\{i,j\}) = V^{\alpha}(\{i,j\}) + \frac{d_k(d_i + d_j)}{\lambda^3}.$$

Thus,

$$V^{\delta}(\cdot) \geq V^{\alpha}(\cdot). \tag{31}$$

Apart from that, from (25), (29) and (26), (30) we have

$$V^{\alpha}(N) = V^{\zeta}(N),$$

$$V^{\alpha}(\{i\}) = V^{\zeta}(\{i\}) + \frac{1}{2\lambda^3}(D_N - d_i)^2,$$

$$V^{\alpha}(\{i,j\}) = V^{\zeta}(\{i,j\}) + \frac{1}{\lambda^3}d_k^2.$$

Thus,

$$V^{\alpha}(\cdot) \geq V^{\zeta}(\cdot). \tag{32}$$

Inequalities (17) and (18) imply that

$$V^{\delta}(\cdot) \geq V^{\alpha}(\cdot) \geq V^{\zeta}(\cdot).$$

5. New characteristic function

Developing the idea of simplification of the technique for calculating c.f.s, we introduce the new definition for the characteristic function

$$V^{\eta}(S,\cdot) = \begin{cases} 0, & S = \{\emptyset\}, \\ \sum\limits_{i\in S} K_i(u_S^*, u_{N\setminus S}^{NE}, \cdot), & S \subset N, \\ \max\limits_{u_1,\ldots,u_n} \sum\limits_{i=1}^{n} K_i(u_1,\ldots,u_n,\cdot), & S = N. \end{cases} \tag{33}$$

where $u^* = \{u_i^*\}_{i\in N}$ is the profile of strategies for which the maximal value of payoff function is achieved for all players, $u_S^* = \{u_i^*\}_{i\in S}$.

In (33) for players from S we use (obtained earlier) strategies u_S^* from optimal profile u^* (as in $\zeta-$c.f.) and for players from $N \setminus S$ we use (obtained earlier) strategies $u_{N\setminus S}^{NE}$ from the Nash equilibrium strategies for all players (as in $\delta-$ c.f.). This provides certain technical advantages as will be reported in a subsequent paper.

6. Conclusion

In this paper we considered three different approaches to the calculation of characteristic function in differential games and applied all of them to differential games with prescribed and random duration. We analyzed the obtained functions and their relations. Also a new way of the characteristic function construction has been introduced.

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