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Two-Stage Network Formation Game with Heterogeneous Players and Private Information*

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Abstract We consider a two-stage network formation game with heterogeneous players and private information. The player set consists of a leader and a finite number of other common players, which are divided into two types, passive and positive players. At the first stage, the leader suggests a connected communication network for all players to join. While it is assumed that the link information which every common player receives from the leader is private. Based on the private information, every player chooses the action, accept or reject, at the second stage. A network is formed finally. We show the existence of subgame perfect Nash equilibrium in the game. The result is illustrated by an example.

Keywords: heterogeneous players, private information, Myerson value, subgame perfect Nash equilibrium.

1. Introduction

In recent years, network games, network stability, network formation as well as issues about communication networks have been widely studied. Jackson and Wolinsky (1996) initially proposed the concept of pairwise stability to characterize stable network in which the rule of network formation is called JW rule. Bala and Goyal (2000) mainly studied the Nash equilibrium network and its dynamic formation process, showing that the Nash network has special structures, such as the star or the wheel. Avrachenkov et al. (2011) addressed network formation issue using cooperative game theory and solve the cooperative network formation game with the Nash bargaining solution concept.

An important extensive research in network formation is to introduce heterogeneity. There are various types of heterogeneity, such as heterogeneous players, heterogeneous costs of forming links, heterogeneous information delivering qualities of links, etc. Heterogeneous players were introduced in (Larrosa and Tohme, 2003) where the payoff of each player is not only associated with the number of links in those paths, the end of which is such player, but also related with the values of himself, and the values of players are various. Galeotti et al. (2006) introduced heterogeneous costs of forming links as well as heterogeneous values in the two-way flow model, and proved that centrality and short average distances between players are robust features of equilibrium networks.

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Besides, Petrosyan and Sedakov (2014) considered the multistage network games with perfect information in which players can change the network structure at each stage, and proposed a method for finding optimal behavior for players in games of this type. The endogenous dynamic formation of the network was introduced in (Aumann and Myerson, 2003) where an auxiliary linking game which consists of pairs of players being offered to form links while the offers are made one by one according to some chosen order of feasible links was constructed. And the linking game was with perfect information.

In practice, heterogeneous people are fairly common among the community, for instance, female and male, individuals with various education backgrounds, like bachelor, master, doctor, people from different countries, and so on. And it is also reasonable that various people have different standards and face various cases although in the same community, such as different levels of salary, getting different information about the community, etc. In this paper, we consider the game of incomplete information, and to simplify the complicated case brought by incomplete information, heterogeneous players are introduced simultaneously.

The paper is organized as follows. In Section 2 some basic definitions and notations are briefly introduced. And in Section 3, the model of two-stage network formation game with heterogeneous players and private information is introduced. Then the two-stage game introduced in Section 3 in extensive form is described in Section 4. Section 5 contains the theorem about the existence of subgame perfect Nash equilibrium in the game as well as a corresponding example.

2. Basic Definitions and Notations

Let the set of players be $N = \{1, \dots, n\}$, $|N| = n \geq 3$. Suppose there is a player called the leader of other players referred as player 1. A cooperative game with transferable utility is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns to every coalition of players $S \subseteq N$ its worth $v(S)$, with $v(\emptyset) = 0$. For simplicity of notation and if no ambiguity appears we write v when we refer to a game (N, v). A singleton solution of game (N, v) is a function $\xi : G \to \mathbb{R}^N$, where G is the set of games (N, v), and $\xi = (\xi_1, \dots, \xi_n)$ is a vector of payoffs to players in v.

A communication structure on N is specified by a graph (N, Γ), where $\Gamma \subseteq \Gamma_N^c = \{ij \mid i, j \in N, i \neq j\}$ is a collection of unordered pairs of nodes. And similarly, we write Γ when we refer to a graph (N, Γ). In graph Γ, a sequence of different nodes $(i_1, \dots, i_k), k \geq 2$ is a path from i_1 to i_k, if for all $h = 1, \dots, k-1, i_h i_{h+1} \in \Gamma$. We say two nodes are connected, if there exists a path from one node to another, and graph Γ is connected, if any two nodes are connected in graph Γ. Here we denote the set of all connected graph on N by $G(N)$.

Given the characteristic function $v(S)$, $S \subseteq N$ and graph Γ, determine the new characteristic function using the following approach:

$$v^{\Gamma}(S) = \sum_{T \in S/\Gamma} v(T), \tag{1}$$

where $S/\Gamma = \{\{i \mid i, j \text{ are connected in } S \text{ by } \Gamma\} \mid j \in S\}$.

Vector $\xi_{\Gamma} = (\xi_1(\Gamma), \dots, \xi_n(\Gamma))$ is defined as a payoff vector in cooperative game with the given graph Γ. For instance, given $v(S)$, $S \subseteq N$ and Γ, if the Myerson value (Myerson, 1977) $Y(\Gamma)$ is chosen as the cooperative solution concept, then

$$\xi_i(\Gamma) = Y_i(\Gamma) = Sh_i(v^\Gamma), \tag{2}$$

for all $i \in N$, where $Sh_i(v^\Gamma)$ is the i-th component of the Shapley value (Shapley, 1953) of player i in game (N, v^Γ).

3. The Model

3.1. Two-Stage Network Formation Game with Private Information

Two-stage network formation game with private information takes places as follows.

Stage 1. The leader chooses a network (graph) Γ from his strategy set U_1 (e.g. he starts a joint project), where U_1 is a given set of connected graphs without loops on N. The cooperative game v showing the power of any coalition S (in the project) is given and known for all players.

After choosing network Γ, the leader informs players $2, \ldots, n$ about the links that the player will have in Γ. The information is private, which means that if player 1 chooses network Γ, then player i will get information from player 1 that he will have the set of links $\Gamma(i) = \{ij \mid ij \in \Gamma\}$ in the network. Therefore, we have the game with imperfect information.

Stage 2. Based on the private information from the leader, players $2, ..., n$ simultaneously and independently choose an action from common strategy set U, which is {accept, reject}. By accepting the network, player $j \in \{2, \ldots, n\}$ joins the network. Otherwise, he starts playing as individual player. If he accepts the network, he pays a fee of $\theta_i(\Gamma)$ which is a function of network Γ. We assume that the network is formed only if all players accept the network simultaneously. Otherwise, the network is not formed and all players act as individual players. If Γ is formed, player i gets a payoff of $\xi_i(\Gamma) - \theta_i(\Gamma)$. We notice that in the following part, we denote the action 'accept' by a, and action 'reject' by r.

3.2. Heterogeneous Players: Passive and Positive

With private information, players are not sure about the network structure which is suggested by player 1 at the first stage. Consequently, every player needs to guess the structure of the network based on the private information, thus choosing the action according to the payoff which is strongly related to the network structure.

According to the private information which players $N \setminus \{1\}$ get, it is easily shown that the set of all the networks player i expects to be formed is $\{\Gamma^i \in G(N) \mid \Gamma(i) \subseteq \Gamma^i \subseteq \Gamma(i) \cup \Lambda\}$, denoted by PI_i, where $\Lambda = \{jk \mid jk \in \Gamma_N^c, j \neq i, k \neq i, j \neq k\}$. Thus player i is able to choose his action based on the payoff $\xi_i(\Gamma^i) - \theta_i(\Gamma^i)$, $\Gamma^i \in PI_i$.

Obviously $\xi_i(\Gamma^i) - \theta_i(\Gamma^i)$ may be different for different $\Gamma^i \in PI_i$. We call player i a *passive player*, if he chooses action based on the payoff $\min\limits_{\Gamma^i \in PI_i} \{\xi_i(\Gamma^i) - \theta_i(\Gamma^i)\}$. The set of passive players in N is denoted by P. On the contrary, player i is called a *positive player*, if he chooses the action according to payoff $\max\limits_{\Gamma^i \in PI_i} \{\xi_i(\Gamma^i) - \theta_i(\Gamma^i)\}$, and we denote the set of positive players in N by Q. And we assume $P \cup Q = N \setminus \{1\}$ holds. Thus, the payoff of player $i \in N \setminus \{1\}$ in the described two-stage game if at the end of stage 2 the network Γ is formed is

$$K^i(\Gamma) = \mathbb{I}\{i \in P\} \cdot \min_{\Gamma^i \in PI_i} \{\xi_i(\Gamma^i) - \theta_i(\Gamma^i)\} + \mathbb{I}\{i \in Q\} \cdot \max_{\Gamma^i \in PI_i} \{\xi_i(\Gamma^i) - \theta_i(\Gamma^i)\} \tag{3}$$

where

$$\mathbb{I}\{i \in S\} = \begin{cases} 1, & i \in S, \\ 0, & i \notin S. \end{cases} \tag{4}$$

While if Γ is not formed, then player i's payoff in the two-stage game is $v\{i\}$.

4. Two-Stage Game as a Game in Extensive Form

The described two-stage game with private information can be regarded as an extensive-form game Φ with player set N on a game tree denoted by Z. Let $X = X_1 \cup \cdots \cup X_n \cup X_{n+1}$ be the finite set of vertices, with $X_1 = \{x_0\}$ being the only personal vertex of player 1, X_i being the set of personal vertices of player $i \in N \setminus \{1\}$ and $X_{n+1} = \{x : F_x = \emptyset\}$ being the set of terminal vertices at which the game ends and players get their payoffs. For any $x \in X$, F_x is the set of those vertices which can be realized immediately after the vertex x has been realized, and $F_x^2 = F(F_x)$, $F_x^k = F(F_x^{k-1})$. By the construction, $F_{x_0} = X_2$ and $\bigcup\limits_{x \in X_i} F_x = X_{i+1}$ for $i = 2, \ldots, n$. Thus we have $|X_1| = 1$, $|X_i| = 2^{i-2}|U_1|$, for $i \in N \setminus \{1\}$, $|X_{n+1}| = 2^{n-1}|U_1|$. Specifically, we denote the vertex to which the game process moves after player 1 suggesting network Γ_k by $x_{\Gamma_k} \in X_2$.

In terms of different private information players get, the set of personal vertices of player $i \in N \setminus \{1\}$ is partitioned into subsets X_i^j, which is referred to as information sets of player i. Specifically, $X_i^j = \{x, y \mid x \in F_{x_{\Gamma_k}}^{i-2},\ y \in F_{x_{\Gamma_g}}^{i-2},\ \Gamma_k(i) = \Gamma_g(i)\}$. For any player $i \in N \setminus \{1\}$ and $x \in X_i$, player i does not know the vertex itself, but knows that this vertex is in a certain information set $X_i^j \subset X_i$.

Now define the strategy of player $i \in N$ in the described two-stage game. A strategy of player 1 is a rule u_1 assigning an action from the set U_1 to the only personal vertex x_0. And the strategy of player $i \in N \setminus \{1\}$ is a rule u_i assigning an action from the action set $\{a, r\}$ to any information set $X_i^j \subset X_i$. And a strategy profile $u = (u_1, \ldots, u_n)$ can uniquely define a terminal vertex, in particular, the terminal vertex which is achieved by the strategy profile $u = (u_1, \ldots, u_n)$, where $u_1(x_0) = \Gamma_k$, is denoted by $x_{\Gamma_k,(u_2,\ldots,u_n)}$. Subgame which begins at vertex x_{Γ_k} is denoted by $Z(x_{\Gamma_k})$, and $u_{ik}, i \in N \setminus \{1\}$ denotes the truncation of strategy u_i to subgame $Z(x_{\Gamma_k})$. In other words, u_{ik} is a rule assigning an action from $\{a, r\}$ to any information set $X_i^j \subset X_i$, $F_{x_{\Gamma_k}}^{i-2} \subseteq X_i^j$.

Given the characteristic function $v(S)$, $S \subseteq N$, the payoff to player $i \in N \setminus \{1\}$ at the terminal vertex $x_{\Gamma_k,(u_2,\ldots,u_n)}$ is defined as

$$K_i(u) = H_i\left(x_{\Gamma_k,(u_2,\ldots,u_n)}\right) = \begin{cases} v(\{i\}), & \exists u_j(X_j^l) = r, F_{x_{\Gamma_k}}^{j-2} \subseteq X_j^l, \\ K^i(\Gamma_k), & \text{otherwise}. \end{cases} \tag{5}$$

And for player 1, it is defined as

$$K_1(u) = H_1\left(x_{\Gamma_k,(u_2,\ldots,u_n)}\right) = \begin{cases} v(\{1\}), & \exists u_j(X_j^l) = r, F_{x_{\Gamma_k}}^{j-2} \subseteq X_j^l, \\ \xi_1(\Gamma_k) - \theta_1(\Gamma_k), & \text{otherwise}. \end{cases} \tag{6}$$

The game proceeds as follows. At vertex x_0, player 1 chooses a network $\Gamma_k \in U_1$, then the game process moves to information sets $X_2^{2'}, \ldots, X_n^{n'}$ simultaneously, where $X_2^{2'} = \{x_{\Gamma_j} \mid \Gamma_j(2) = \Gamma_k(2)\}$, $F_{x_{\Gamma_k}}^{i-2} \subseteq X_i^{i'}$, $i = 3, \ldots, n$, and players $N \setminus \{1\}$ choose actions from $\{a, r\}$ for the corresponding information sets independently, with $u_i(X_i^{i'})$ being the choice of player i. Finally, the game terminates at vertex $x_{\Gamma_k,(u_2,\ldots,u_n)}$, and player $i \in N$ gets his payoff $H_i\left(x_{\Gamma_k,(u_2,\ldots,u_n)}\right)$ defined by (5) and (6).

5. Main Result and Example

Theorem 1. *The extensive-form game Φ on game tree Z admits a subgame perfect Nash equilibrium (SPNE).*

Proof. Consider the families of subgame $Z(x_{\Gamma_k})$, $\Gamma_k \in U_1$, there are only two kinds of payoff vectors among all the terminal vetices. The first case is when all players $N \setminus \{1\}$ choose action a for the corresponding information sets. And the other case is when there exists at least one player choosing action r. Thus, it is easily known that $(u^*_{2k}, \ldots, u^*_{nk})$, where $u^*_{ik}(X_i^p) = r$, $i \in N \setminus \{1\}$ is the Nash equilibrium of subgame $Z(x_{\Gamma_k})$, $\Gamma_k \in U_1$ because any player can not change the payoff in subgame $Z(x_{\Gamma_k})$ by deviating from choice r. Therefore, strategy profiles $(\Gamma_l, u^*_2, \ldots, u^*_n)$, where $u^*_i = (u^*_{i1}, \ldots, u^*_{i|U_1|})$, $l = 1, \ldots, |U_1|$, $i \in N \setminus \{1\}$ are all subgame perfect Nash equilibria of the game. The theorem is proved. $\square$

Example 1. Let the set of players be $N = \{1, 2, 3\}$. The values of characteristic function are $v(\{1\}) = 1$, $v(\{2\}) = v(\{3\}) = 1/2$, $v(\{1,2\}) = 3$, $v(\{1,3\}) = 2$, $v(\{2,3\}) = 3/2$, $v(\{1,2,3\}) = 5$. The strategy set of the leader (player 1) is $U_1 = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$, where $\Gamma_1 = \{12, 13\}$, $\Gamma_2 = \{12, 13, 23\}$, $\Gamma_3 = \{12, 23\}$, $\Gamma_4 = \{13, 23\}$.

Here we use Myerson value (Myerson, 1977) as a singleton solution, and $\theta_i(\Gamma) = c|\Gamma(i)|$ is defined as the cost for player i to hold links in network Γ, where $|\Gamma(i)|$ is the number of links in $\Gamma(i)$, c being the holding cost per link.

Fig. 1 to Fig. 4 show the game trees and SPNE with $c = 1/2$, and different cases: 1) $Q = \{2\}, P = \{3\}$; 2) $Q = \{2, 3\}$; 3) $P = \{2\}, Q = \{3\}$; 4) $P = \{2, 3\}$ respectively. Fig. 5 to Fig. 8 show the game trees and SPNE with $c = 1/4$, and cases: 1) $-$ 4) respectively. And the colored links in figures show the SPNE (not unique) in the game.

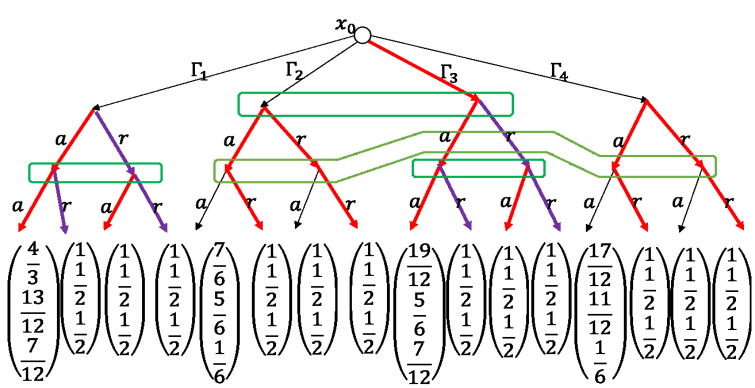

Fig. 1. Two-stage game with $c = 1/2, Q = \{2\}, P = \{3\}$.

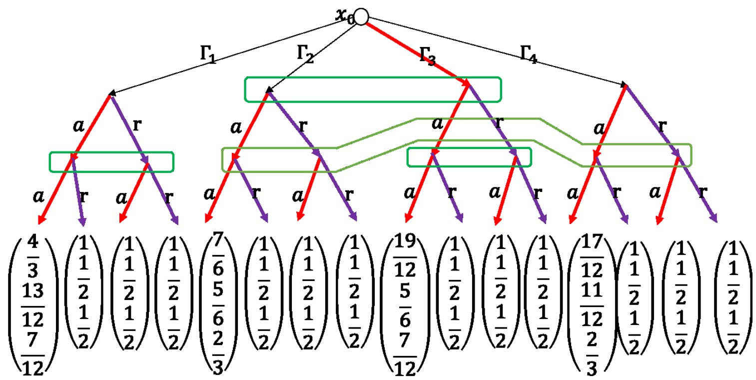

Fig. 2. Two-stage game with $c = 1/2, Q = \{2, 3\}$.

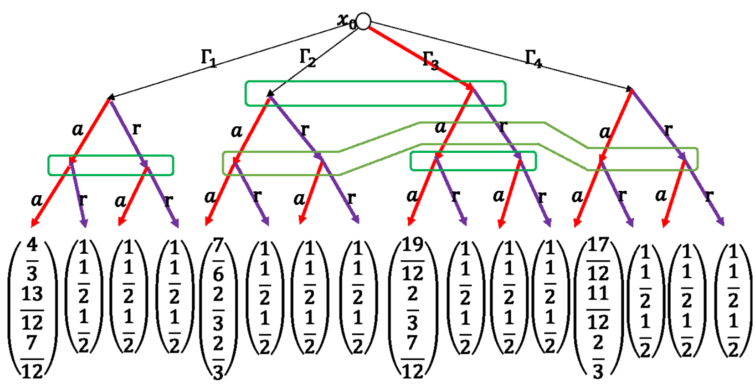

Fig. 3. Two-stage game with $c = 1/2, P = \{2\}, Q = \{3\}$.

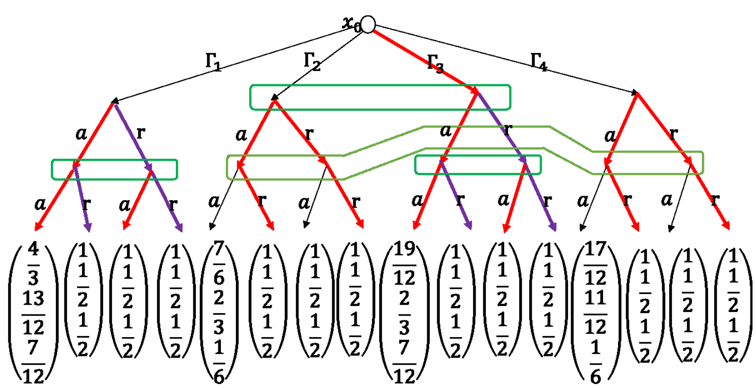

Fig. 4. Two-stage game with $c = 1/2, P = \{2, 3\}$.

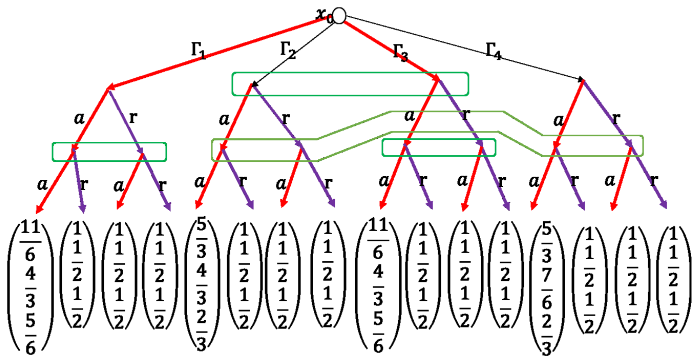

Fig. 5. Two-stage game with $c = 1/4, Q = \{2\}, P = \{3\}$.

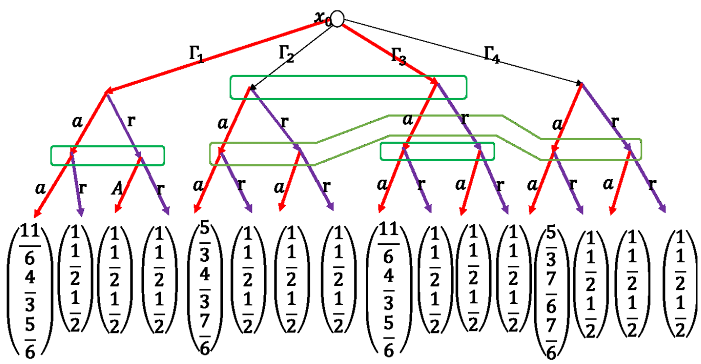

Fig. 6. Two-stage game with $c = 1/4, Q = \{2, 3\}$.

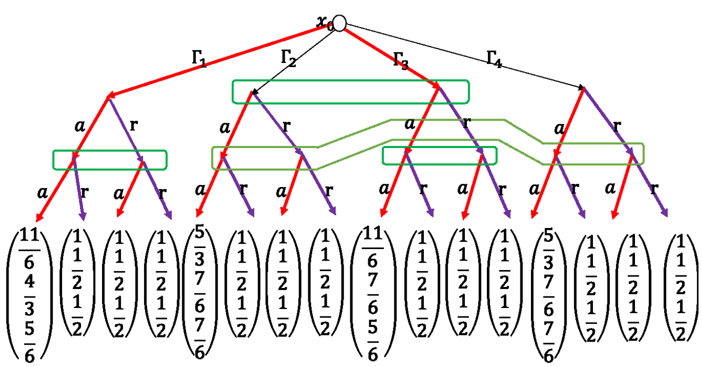

Fig. 7. Two-stage game with $c = 1/4, P = \{2\}, Q = \{3\}$.

E.g., in Fig. 1 in subgame $Z(x_{\Gamma_1})$, with payoff vectors shown at the terminal vertices, it is easily seen that both the strategy profiles (a, a) and (r, r) are the Nash equilibria. Then in subgame $Z(x_{\Gamma_2})$, both the strategy profiles (a, r) and (r, r) are

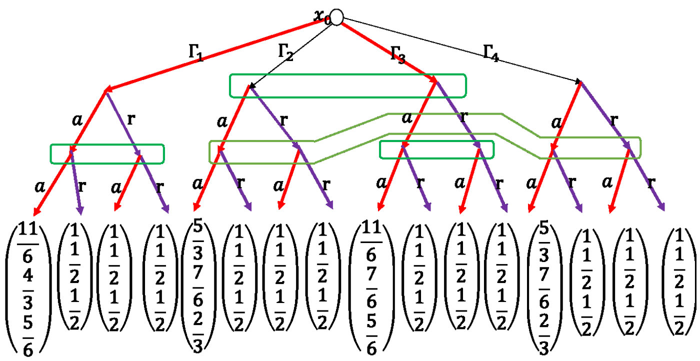

Fig. 8. Two-stage game with $c = 1/4, P = \{2, 3\}$.

the Nash equilibria. And in subgame $Z(x_{\Gamma_3})$, both the strategy profiles (a, a) and (r, r) are the Nash equilibria. In subgame $Z(x_{\Gamma_4})$, both the strategy profiles (a, r) and (r, r) are the Nash equilibria. Finally, in game Z with x_0 as the initial vertex, we can get that all the strategy profiles shown in the Table 1 are the SPNE of the game.

Table 1. All SPNE and corresponding networks in the game described in Fig. 1.

Strategy Profiles \ Players / Networks	Player 1	Player 2	Player 3
Γ_3	Γ_3	(a, a, a)	(a, r, a)
	Γ_3	(a, a, r)	(a, r, a)
	Γ_3	(r, a, a)	(r, r, a)
	Γ_3	(r, a, r)	(r, r, a)
Γ_1	Γ_1	(a, r, r)	(a, r, r)
	Γ_1	(a, r, a)	(a, r, r)
$\emptyset$	Γ_1	(r, r, r)	(r, r, r)
	Γ_2	(r, r, r)	(r, r, r)
	Γ_3	(r, r, r)	(r, r, r)
	Γ_4	(r, r, r)	(r, r, r)
	Γ_1	(r, r, a)	(r, r, r)
	Γ_2	(r, r, a)	(r, r, r)
	Γ_3	(r, r, a)	(r, r, r)
	Γ_4	(r, r, a)	(r, r, r)

We can also analyze the set of SPNE for games from Fig. 2 to Fig. 8 respectively. And in fact, the sets of SPNE in games from Fig. 1 to Fig. 4 are not the same. For instance, strategy profile (Γ_3, u_2^*, u_3^*) where $u_2^* = (r, a, a)$, $u_3^* = (r, a, a)$ is a SPNE in the game shown in Fig. 3. While it is not a SPNE in the game described in Fig. 4. Thus, we can conclude that the types of players can have an effect on SPNE. While it is also easily seen that, for games which are shown from Fig. 5 to Fig. 8, the sets of SPNE are the same. In other word, the types of players do not affect the the set of SPNE in those cases.

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