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Acceptable Points in Antagonistic Games with Ordered Outcomes

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Abstract The acceptability concept is a naturally generalization of the equilibrium concept. An outcome of a game is called an acceptable one if no players which have an objection to it in the form their strategies. Note that for the class of games with payoff function, acceptability condition is equivalent to individual rationality condition. This article is a continuation of the previous work of the author (see Rozen, 2018). The aim of the article is a detection of structure of the set of acceptable outcomes in antagonistic games with ordered outcomes (all required definitions for antagonistic games with ordered outcomes indicated in the introduction, see section 1). In section 2 we offer some classification for outcomes in antagonistic games. Using this classification, a localization of acceptable outcomes is specified (see section 3). In section 4 certain sufficient conditions for non-emptiness and uniqueness of acceptable outcomes are found. Some examples related to localization of acceptable outcomes in antagonistic games with ordered outcomes are given.

Keywords: antagonistic game with ordered outcomes, acceptable point, saddle point, centre of game, periphery of game.

1. Introduction

1.1. Games with ordered outcomes

A game of n players with ordered outcomes in the normal form can be given as a system of the type

$$G = \langle N, (X_i)_{i \in I}, A, (\omega_i)_{i \in I}, F \rangle \ . \tag{1}$$

where $N = \{1, \ldots, n\}$ is a set of *players*, $n \geq 2$; X_i is a *set of strategies* of the player i; A is a *set of outcomes*; $\omega_i \subseteq A^2$ is an order relation which represents the preferences of player i; F is *a realization function*, i.e. a mapping from the set of all situations $X = \prod_{i \in N} X_i$ into the set of outcomes A. For the class of games with ordered outcomes of the type (1), the most important optimality concept is Nash equilibrium.

Definition 1. A situation $x^0 = \left(x_i^0\right)_{i \in N}$ in game G of the form (1) is called *Nash equilibrium point* if for all $i \in N$ and $x_i' \in X_i$ the correlation holds:

$$F\left(x^0 \parallel x_i'\right) \overset{\omega_i}{\leq} F\left(x^0\right). \tag{2}$$

We now consider a concept of *acceptable outcome* for game G of the form (1). Fix some $i \in N$ and put $X_{N \setminus i} = \prod\limits_{\substack{j \in N \\ j \neq i}} X_j$. Note that we can consider $X_{N \setminus i}$ as a set of strategies of the complementary coalition $N \setminus i$. A pair $\left(x_i, x_{N\setminus i}\right)$ where $x_i \in X_i$ and $x_{N\setminus i} \in X_{N\setminus i}$ uniquely defines some outcome in game G which is denoted by $F\left(x_i, x_{N\setminus i}\right)$.

Definition 2. We say that a strategy $x_i^0 \in X_i$ is an *objection of player i to outcome* $a \in A$ if for any strategy $x_{N\setminus i} \in X_{N\setminus i}$ of the complementary coalition the correlation $F\left(x_i^0, x_{N\setminus i}\right) \overset{\omega_i}{>} a$ holds. An outcome $a \in A$ is called *an acceptable one* for player i if he has not objections to it. An outcome a is called *acceptable one in game G* if this outcome is acceptable for all players $i \in N$.

Therefore an outcome $a \in A$ is an acceptable one in game G if for any $i \in N$ and $x_i \in X_i$ there exists a strategy $x_{N\setminus i} \in X_{N\setminus i}$ of the complementary coalition such that the condition $\neg\left(F\left(x_i, x_{N\setminus i}\right) \overset{\omega_i}{>} a\right)$ holds. Indicated strategy $x_{N\setminus i}$ of complementary coalition is called *a punishing strategy*.

Definition 3. An outcome $a \in A$ is called *quite acceptable one* for player i if there exists a strategy $x_{N\setminus i} \in X_{N\setminus i}$ of complementary coalition such that for any $x_i \in X_i$ the condition $\neg\left(F\left(x_i, x_{N\setminus i}\right) \overset{\omega_i}{>} a\right)$ holds. An outcome a is called *quite acceptable one in game G* if it is quite acceptable for all players $i \in N$.

These concepts are transferred from outcomes of game G to its situations. Namely, a situation $x \in X$ in game G is called *acceptable* (or *quite acceptable*) *one* if the outcome $F(x)$ is acceptable (or quite acceptable) respectively.

Remark 1. Nash equilibrium point is a quite acceptable (and hence an acceptable also) point in game G with ordered outcomes.

1.2. Antagonistic games with ordered outcomes

An antagonistic game with ordered outcomes is a game of the type (1) in which a number of players is equal two and their preference relations are mutually inverse. We consider such a game in the form

$$G = \langle X, Y, A, \omega, F\rangle \tag{3}$$

where X is a set of strategies of a player 1, Y is a set of strategies of a player 2, A is a set of outcomes, ω is a (partial) order relation on the set A, $F\colon X \times Y \to A$ is a realization function. The preferences of the player 1 are given by the order ω and preferences of player 2 are given by the inverse order ω^{-1}. We assume that $|X| \geq 2$, $|Y| \geq 2$, $|A| \geq 2$. For antagonistic games, definitions 1, 2 and 3 take the following form.

Definition 4. A situation (x_0, y_0) in game G of the form (3) is called *a saddle point* if for any $x \in X, y \in Y$ hold the correlations

$$F(x, y_0) \overset{\omega}{\leq} F(x_0, y_0) \overset{\omega}{\leq} F(x_0, y). \tag{4}$$

Next using definition 2, we obtain that in game G of the form (3), an outcome $a \in A$ is acceptable for player 1 if the following formula holds:

$$\neg(\exists x \in X)(\forall y \in Y)\, F(x, y) \overset{\omega}{>} a, \text{ i.e. } (\forall x \in X)(\exists y \in Y)\neg\left(F(x, y) \overset{\omega}{>} a\right).$$

Dually an outcome $a \in A$ is acceptable for player 2 if the formula

$$(\forall y \in Y)\,(\exists x \in X)\,\neg\left(F(x,y) \overset{\omega}{<} a\right)$$

holds. Thus an arbitrary outcome $a \in A$ is acceptable in game G of the form (3) if the following system of conditions

$$\begin{cases} (\forall x \in X)\,(\exists y \in Y) & \neg\left(F(x,y) \overset{\omega}{>} a\right) \\ (\forall y \in Y)\,(\exists x \in X) & \neg\left(F(x,y) \overset{\omega}{<} a\right) \end{cases} \tag{5}$$

holds. Further, an outcome $a \in A$ is quite acceptable in antagonistic game G of the form (3) if the system of conditions

$$\begin{cases} (\exists y \in Y)\,(\forall x \in X) & \neg\left(F(x,y) \overset{\omega}{>} a\right) \\ (\exists x \in X)\,(\forall y \in Y) & \neg\left(F(x,y) \overset{\omega}{<} a\right) \end{cases} \tag{6}$$

holds. Remark that the system (6) can be obtained from the system (5) by nontrivial permutation of unlike quantifiers. Moreover, strategies $x \in X$ and $y \in Y$ standing in (6) under external quantifiers are punishing strategies of players 1 and 2, respectively.

1.3. Acceptable and quite acceptable outcomes in games with payoff functions

Consider an antagonistic game with payoff function $\Gamma = \langle X, Y, u\rangle$ where X is a set of strategies of player 1, Y is a set of strategies of player 2, $u(x,y)$ is a payoff function. We can mean a game Γ as a game with ordered outcomes, in which the set of strategies of players are the same, a set of outcomes is real numbers $\mathbb{R}$, realization function is the function $u(x,y)$, and order relation ω is determined by the value of payoff. Put $v_1 = \sup\limits_{x\in X}\inf\limits_{y\in Y} u(x,y)$ be the *lower value* and $v_2 = \inf\limits_{y\in Y}\sup\limits_{x\in X} u(x,y)$ the *upper value* of game Γ. Consider now the following condition.

(C) If the external extremum in $\sup\limits_{x\in X}\inf\limits_{y\in Y} u(x,y)$ *is realized at the point* $x_0 \in X$ *then the inner extremum in* $\inf\limits_{y\in Y} u(x_0,y)$ *must be realized at certain point* $y_0 \in Y$.

It is easy to show that for game Γ considered as a game with ordered outcomes, the set of all acceptable outcomes for player 1 is the interval (v_1,∞) and possibly the point v_1. Moreover, the outcome v_1 is an acceptable one for player 1 if and only if the condition (C) holds. For finding of all acceptable outcomes for player 2 we can use the dual condition (C^*). Thus the set $Ac\,\Gamma$ consisting of all acceptable outcomes of game Γ is the interval (v_1,v_2) and possibly points v_1 and v_2. In particular let the sets X, Y be compact topological spaces and the function u is continuous on $X \times Y$. Then the conditions (C) and (C^*) hold, hence in this case we obtain $Ac\,\Gamma = [v_1,v_2]$. Hence for this game the set of all acceptable outcomes coincides with the set of individual-rational outcomes.

We now consider the quite acceptability conditions for antagonistic game with payoff function $\Gamma = \langle X, Y, u\rangle$ of general form. In accordance with (6), an arbitrary outcome $a \in A$ is quite acceptable in game Γ if and only if the following system holds:

$$\begin{cases} (\exists y \in Y)\,(\forall x \in X) & u(x,y) \le a \\ (\exists x \in X)\,(\forall y \in Y) & u(x,y) \ge a \end{cases} \tag{7}$$

Denoting standing under external quantifiers in (7) punishing strategies of players 1 and 2 by x_0 and y_0 respectively, we obtain for any $x \in X$ and $y \in Y$ the double inequality $u(x, y_0) \leq a \leq u(x_0, y)$. Setting $x = x_0$ and $y = y_0$ we have $a = u(x_0, y_0)$, hence the situation (x_0, y_0) is a saddle point in pure strategies for game Γ. Then the game Γ has a value v and $a = v$. We show that in antagonistic game with payoff function the existence of quite acceptable outcome is equivalent to existence of saddle point and in this case the quite acceptable outcome coincides with value of the game.

Therefore, for game with payoff function the set of acceptable (and also quite acceptable) points has a very simple structure. However, for antagonistic games with order outcomes, the structure of acceptable points is more complex. A study of these problems is carried out in the following sections.

2. Classification of outcomes in antagonistic games

As a basis for the classification of the set of outcomes for antagonistic game with ordered outcomes we take the non-forbidden outcomes of its players. Recall that the sets of non-forbidden outcomes for players 1 and 2 are respectively

$$U(1) = \{a \in A\colon (\forall y \in Y)(\exists x \in X)\left(F(x, y) \overset{\omega}{\geq} a\right)\}, \tag{8}$$

$$U(2) = \{a \in A\colon (\forall x \in X)(\exists y \in Y)\left(F(x, y) \overset{\omega}{\leq} a\right)\}. \tag{9}$$

Note that $U(1) \cup U(2)$ and $(U(1) \cup U(2))'$ form a partition of the set all outcomes A; in addition $U(1) \triangle U(2)$ and $U(1) \cap U(2)$ form a partition of $U(1) \cup U(2)$ (here the sign $\triangle$ denotes the symmetric difference of sets). Therefore in antagonistic game G of the form (3), the set A of all outcomes can be represented as a disjoint union of the following three subsets:

$$A = (U(1) \cap U(2)) \cup (U(1) \triangle U(2)) \cup (U(1) \cup U(2))'. \tag{10}$$

We now introduce the designations for antagonistic game G of the form (3):

$$\begin{aligned} &Z(G) = U(1) \cap U(2) \quad \text{called } \textit{the centre of game } G; \\ &D(G) = U(1) \cup U(2) \quad \text{called } \textit{the domain of game } G; \\ &R(G) = U(1) \triangle U(2) \quad \text{called } \textit{the ring of game } G; \\ &P(G) = (U(1) \cup U(2))' \quad \text{called } \textit{the periphery of game } G. \end{aligned} \tag{11}$$

Hence in according to (10), the set A of all outcomes in antagonistic game G is disjoint union of the centre, the ring, and the periphery:

$$A = Z(G) \cup R(G) \cup P(G). \tag{12}$$

In addition the domain of game G is a disjoint union its centre and ring, i.e. $D(G) = Z(G) \cup R(G)$.

Example 1. Consider an antagonistic game G with ordered outcomes which has a saddle point (x_0, y_0). In this case the centre of game G consists of the one element $F(x_0, y_0)$; the ring of game G consists of outcomes comparable with $F(x_0, y_0)$ under order ω (except for the outcome $F(x_0, y_0)$ itself) and the periphery consists of outcomes incomparable with $F(x_0, y_0)$ under order ω.

3. Localization of acceptable outcomes in antagonistic games

3.1. The centre and the periphery of a game consist of acceptable outcomes

Using introduced above classification of outcomes for antagonistic games with ordered outcomes, we establish the following result, connected with localization of the acceptable outcomes in such games.

Theorem 1. *Let $G = \langle X, Y, A, \omega, F \rangle$ be an antagonistic game with ordered outcomes. Then*

1. *All outcomes of its centre are acceptable;*
2. *All outcomes of its periphery are acceptable.*

Proof (of theorem 1). Denote by $V^*(k)$ the set of all outcomes in game G which are strictly guaranteed to the player $k = 1, 2$:

$$V^*(1) = \{a \in A\colon\ (\exists x \in X)(\forall y \in Y)\left(F(x,y) \overset{\omega}{>} a\right)\}, \tag{13}$$

$$V^*(2) = \{a \in A\colon\ (\exists y \in Y)(\forall x \in X)\left(F(x,y) \overset{\omega}{<} a\right)\}. \tag{14}$$

Lemma 1. *For any $a \in V^*(1)$ and $b \in U(2)$ holds $a \overset{\omega}{<} b$.*

Proof (of lemma 1). The condition $a \in V^*(1)$ means that $F(x_0, y) \overset{\omega}{>} a$ for certain strategy $x_0 \in X$ and all $y \in Y$; and the condition $b \in U(2)$ means that the formula $(\forall x \in X)(\exists y \in Y)\left(F(x,y) \overset{\omega}{\leq} b\right)$ holds. Then for $x = x_0$ there exists a strategy $y_0 \in Y$ such that $F(x_0, y_0) \overset{\omega}{\leq} b$. On the other hand setting $y = y_0$, we have the correlation $F(x_0, y_0) \overset{\omega}{>} a$. Thus we obtain the double inequality $a \overset{\omega}{<} F(x_0, y_0) \overset{\omega}{\leq} b$ hence $a \overset{\omega}{<} b$ and Lemma 1 is proved. □

Using lemma 1 we obtain the following equalities:

$$V^*(1) \cap U(2) = \emptyset, \tag{15}$$

$$V^*(2) \cap U(1) = \emptyset. \tag{16}$$

Indeed, assume that $a \in V^*(1) \cap U(2)$ then according to lemma 1 we obtain $a \overset{\omega}{<} a$ that is false. Thus the equality (15) holds and dually we have (16).

Remark that the equality (15) is equivalent to the inclusion $U(2) \subseteq (V^*(1))'$. Dually we have the inclusion $U(1) \subseteq (V^*(2))'$. Now in accordance with (5) the set $Ac\ G$ of all acceptable outcomes in game G can be represented in the form

$$Ac\ G = (V^*(1))' \cap (V^*(2))'. \tag{17}$$

Then using (11), (15), (16). (17) we obtain

$$Z(G) = U(1) \cap U(2) \subseteq (V^*(2))' \cap (V^*(1))' = Ac\ G.$$

and the statement 1 of theorem 1 is proved. To prove the statement 2 of theorem 1 note that $V^*(1) \subseteq U(1)$ and $V^*(2) \subseteq U(2)$, then $V^*(1) \cup V^*(2) \subseteq U(1) \cup U(2)$ hence using (11), (17) we obtain

$$P(G) = (U(1) \cup U(2))' \subseteq (V^*(1) \cup V^*(2))' = (V^*(1))' \cap (V^*(2))' = Ac\ G,$$

which was to be proved. □

Now using (12) we have the corollary.

Corollary 1. *The set $Ac\ G$ of acceptable outcomes in antagonistic game G with ordered outcomes can be represented as disjoint union its centre $Z(G)$, periphery $P(G)$ and acceptable outcomes its ring $R(G)$. At the same time some of these sets (or all these sets) can be empty.*

3.2. Acceptable outcomes in antagonistic game with linearly ordered outcomes

We now show that in antagonistic game G with linearly ordered outcomes, the set of all acceptable outcomes coincides with centre of the game: $Ac\ G = Z(G)$.

Indeed, since in this case for any outcomes $a_1, a_2 \in A$ the following equivalence

$$\neg\left(a_1 \overset{\omega}{>} a_2\right) \Leftrightarrow a_1 \overset{\omega}{\leq} a_2$$

holds, we get the following chain of equivalences for any $a \in A$:

$$\begin{aligned} &a \in (V^*(1))' \Leftrightarrow \neg(a \in (V^*(1))) \Leftrightarrow \neg(\exists x \in X)(\forall y \in Y)\, F(x,y) \overset{\omega}{>} a \Leftrightarrow \\ &\Leftrightarrow (\forall x \in X)(\exists y \in Y)\neg\left(F(x,y) \overset{\omega}{>} a\right) \Leftrightarrow (\forall x \in X)(\exists y \in Y)\, F(x,y) \overset{\omega}{\leq} a \Leftrightarrow \\ &\Leftrightarrow a \in U(2) \end{aligned}$$

Thus, for game G we have the equality $(V^*(1))' = U(2)$ and also the dual equality $(V^*(2))' = U(1)$. Then using these equalities and (17), we obtain

$$Ac\ G = (V^*(1))' \cap (V^*(2))' = U(2) \cap U(1) = U(1) \cap U(2) = Z(G)$$

which was to be proved.

3.3. Example: all acceptable outcomes are in a ring of a game

According to theorem 1, any antagonistic game G with ordered outcomes in which its centre or periphery non-empty, has acceptable outcomes. However, it is possible when both the center and the periphery in game G are empty but the game has acceptable outcomes; in this case in accordance with (12) the set of all outcomes of game G coincides with its ring $R(G)$ and all acceptable outcomes are in a ring of a game. Indeed, consider the following example.

Example 2. Let G be an antagonistic game with ordered outcomes whose realization function F is given by the table 1 and order relation ω on the set of outcomes $A = \{a, b, c, d, e, f\}$ by diagram in pic. 1. Let us construct the extended table of the realization function (see table 2).

Table 1. Realization function F

F	y_1	y_2	y_3
x_1	b	d	f
x_2	e	c	f
x_3	c	b	a

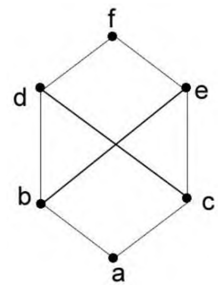

Fig. 1. Order relation ω

Explanation. In table 2 the sign $\left(V^1_{x_i}\right)^*$ denotes the set of all outcomes in game G which are strict guaranteed to player 1 by its strategy $x_i\,(i = 1, 2, 3)$; the sign $\left(V^2_{y_j}\right)^*$ denotes the set of all outcomes which are strict guaranteed to player 2 by its strategy $y_j\,(j = 1, 2, 3)$; the sign $V^*(1)$ denotes the set of all strict guaranteed outcomes of player 1 and $V^*(2)$ the set of all strict guaranteed outcomes of player 2 (see (13), (14)).

Table 2. Extended table for realization function F

F	y_1	y_2	y_3	$V^*(1) = \{a\}$	$U(2) = \{d, e, f\}\ \vert^{\cap}$
x_1	b	d	f	$\left(V^1_{x_1}\right)^* = \{a\}$	$U^2_{x_1} = \{b, d, e, f\}\ \vert^{\uparrow}$
x_2	e	c	f	$\left(V^1_{x_2}\right)^* = \{a\}$	$U^2_{x_2} = \{c, d, e, f\}\ \vert^{\uparrow}$
x_3	c	b	a	$\left(V^1_{x_3}\right)^* = \emptyset$	$U^2_{x_3} = A\ \vert^{\uparrow}$
$U(1) = \{a, b, c\}\ \vert^{\cap}$	$U^1_{y_1} = \{a, b, c, e\}\ \vert^{\downarrow}$	$U^1_{y_2} = \{a, b, c, d\}\ \vert^{\downarrow}$	$U^1_{y_3} = A\ \vert^{\downarrow}$		
$V^*(2) = \{f\}$	$\left(V^2_{y_1}\right)^* = \{f\}$	$\left(V^2_{y_2}\right)^* = \{f\}$	$\left(V^2_{y_3}\right)^* = \emptyset$		

In accordance with (17), we find the set $Ac\ G$ of all acceptable outcomes in the game

$$Ac\ G = (V^*(1))' \cap (V^*(2))' = (V^*(1) \cup V^*(2))' = \{b, c, d, e\}.$$

On the other hand, because $U(1) = \{a, b, c\}$, $U(2) = \{d, e, f\}$, we obtain

$$Z(G) = U(1) \cap U(2) = \{a, b, c\} \cap \{d, e, f\} = \emptyset,$$
$$P(G) = (U(1) \cup U(2))' = (\{a, b, c\} \cup \{d, e, f\})' = A' = \emptyset.$$

Thus in game G of example 2 both the center $Z(G)$ and the periphery $P(G)$ are empty, however the set of acceptable outcomes is non-empty.

3.4. Example: the set of acceptable outcomes in game coincides with its periphery

Another extreme case of the antagonistic game with ordered outcome is possible, when the set of all acceptable outcomes in game G coincides with its periphery $P(G)$.

Example 3. Consider antagonistic game G with ordered outcome whose realization function by the table 3 is given and the ordered set of outcomes by the diagram (Fig. 2).

Table 3. Realization function F

F	y_1	y_2	y_3
x_1	d	c	d
x_2	b	b	c
x_3	d	b	b

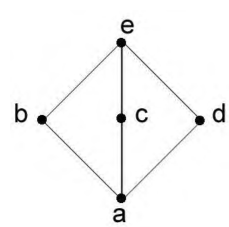

Fig. 2. Order relation ω

The extended table of the realization function is shown in table 4 (in this table symbol designations as in table 2).

Table 4. Realization function F

F	y_1	y_2	y_3	$V^*(1)=\{a\}$	$U(2)=\{e\}$ $\vert^{\cap}$
x_1	d	c	d	$(V^1_{x_1})^*=\{a\}$	$U^2_{x_1}=\{c,d,e\}$ $\vert^{\uparrow}$
x_2	b	b	c	$(V^1_{x_2})^*=\{a\}$	$U^2_{x_2}=\{b,c,e\}$ $\vert^{\uparrow}$
x_3	d	b	d	$(V^1_{x_3})^*=\{a\}$	$U^2_{x_3}=\{b,d,e\}$ $\vert^{\uparrow}$
$U(1)=\{a\}$ $\vert^{\cap}$	$U^1_{y_1}=\{a,b,d\}$ $\vert^{\downarrow}$	$U^1_{y_2}=\{a,b,c\}$ $\vert^{\downarrow}$	$U^1_{y_3}=\{a,c,d\}$ $\vert^{\downarrow}$		
$V^*(2)=\{e\}$	$(V^2_{y_1})^*=\{e\}$	$(V^2_{y_2})^*=\{e\}$	$(V^2_{y_3})^*=\{e\}$		

Using (17), we find the set $Ac\ G$ of all acceptable outcomes in the game

$$Ac\ G = (V^*(1))' \cap (V^*(2))' = (V^*(1) \cup V^*(2))' = \{a, e\}' = \{b, c, d\}.$$

We now find the periphery $P(G)$ of game G:

$$P(G) = (U(1) \cup U(2))' = (\{a\} \cup \{e\})' = \{b, c, d\}.$$

Thus in the game G of example 3 the set of all acceptable outcomes coincides with its periphery.

Remark 2. For any antagonistic game G with ordered outcomes in which the set of all acceptable outcomes coincides with its periphery, the following equalities hold:

$$a) Z(G) = \emptyset; \quad b) V^*(1) = U(1); \quad c) V^*(2) = U(2).$$

Indeed, it follows from the corollary 1 that the equality a) holds. In addition note that in such a game there are no saddle points since the outcome at the saddle point is in centre of the game (see example 1). Let us prove b). Because the evident inclusion $V^*(1) \subseteq U(1)$ holds, it is sufficiently to show the equality $U(1) \setminus V^*(1) = \emptyset$. Assume $a \in U(1) \setminus V^*(1)$, i.e. $a \in U(1)$ and $a \in (V^*(1))'$. From the condition $a \in U(1)$ it follows that $a \in (V^*(2))'$ (see (16)), i.e. outcome a is an acceptable one for player 2; since $a \in (V^*(1))'$ then outcome a is an acceptable one for player 1. Hence the outcome a is an acceptable one in game G and according to our assumption we obtain $a \in P(G)$. On the other hand since

$$P(G) = (U(1) \cup U(2))' = (U(1))' \cap (U(2))'.$$

the inclusion $P(G) \subseteq (U(1))'$ holds, then $a \in (U(1))'$ which contradicts to assumption $a \in U(1) \setminus V^*(1)$. The equality b) is proved: the equality c) is established dually. Using conditions a), b), c), we obtain that in game G the pair of subsets $\{V^*(1), V^*(2)\}$ forms a partition of the domain $D(G)$.

Note that all conditions indicated in remark 2 are true for example 3.

4. Conditions for non-emptiness and uniqueness of acceptable outcomes

4.1. Sufficient condition for non-emptiness of acceptable outcomes concerning the ordered set of outcomes

Theorem 2. *Assume in antagonistic game $G = \langle X, Y, A, \omega, F \rangle$ with ordered outcomes all chains of ordered set $\langle A, \omega \rangle$ are finite. Then the set $Ac\ G$ of acceptable outcomes in game G is non-empty.*

Proof (of theorem 2). Case 1. $V^*(1) \neq \emptyset$, $V^*(2) \neq \emptyset$. Since by assumption all chains in the ordered set $\langle A, \omega \rangle$ are finite, both ascending chain condition and descending chain condition are hold, hence every non-empty subset in A has a maximal and minimal element (see Birkhoff, 1964). Let a^* be a maximal element in non-empty subset $V^*(1)$ and b^* a minimal element in non-empty subset $V^*(2)$. Because $a^* \in V^*(1)$ then there exists a strategy $x_0 \in X$ which strictly guarantees the outcome a^* to player 1, i.e.

$$(\forall y \in Y) \quad F(x_0, y) \overset{\omega}{>} a^*. \tag{18}$$

Dually, because $b^* \in V^*(2)$ then there exists a strategy $y_0 \in Y$ which strictly guarantees the outcome b^* to player 2, i.e.

$$(\forall x \in X) \quad F(x, y_0) \overset{\omega}{<} b^*. \tag{19}$$

Setting $y = y_0$ in (18) and $x = x_0$ in (19), we obtain $a^* \overset{\omega}{<} F(x_0, y_0) \overset{\omega}{<} b^*$. Since a^* is a maximal element in $V^*(1)$ and b^* is a minimal element in $V^*(2)$, it follows from the last double inequality that $F(x_0, y_0) \notin (V^*(1) \cup V^*(2))$; according to (17) we get $F(x_0, y_0) \in (V^*(1) \cup V^*(2))' = (V^*(1))' \cap (V^*(2))' = Ac\, G$. Thus the outcome $F(x_0, y_0)$ is an acceptable one in game G.

Case 2. $V^*(1) \neq \emptyset$, $V^*(2) = \emptyset$. Like the case 1 we get that there exists a maximal element a' in non-empty set $V^*(1)$. Since $a' \in V^*(1)$ there exists a strategy $x' \in X$ which strictly guarantees the outcome a^* to player 1, i.e.

$$(\forall y \in Y) \quad F(x', y) \overset{\omega}{>} a'. \tag{20}$$

Fix arbitrarily a strategy $y' \in Y$ then according to (20) we obtain $F(x', y') \overset{\omega}{>} a'$. Moreover because element a' is a maximal one in subset $V^*(1)$, from the last correlation we have $F(x', y') \notin V^*(1)$, i.e. the outcome $F(x', y')$ is an acceptable one for player 1. Since by assumption $V^*(2) = \emptyset$ any outcome in game G, in particular, the outcome $F(x', y')$ is an acceptable one for player 2. Thus the outcome $F(x', y')$ is an acceptable one in game G.

Case 3. $V^*(1) = \emptyset$, $V^*(2) \neq \emptyset$. The proof is dual to the case 2.

Case 4. $V^*(1) = \emptyset$, $V^*(2) = \emptyset$. In this case any outcome $a \in A$ is acceptable in game G. Theorem 2 is proved. □

Since in finite ordered set all chains are finite, we have

Corollary 2. *Let $G = \langle X, Y, A, \omega, F \rangle$ be an antagonistic game with ordered outcomes in which the set A of outcomes is finite. Then the set $Ac\, G$ of acceptable outcomes in game G is non-empty.*

Corollary 3. *Let $G = \langle X, Y, A, \omega, F \rangle$ be an antagonistic game with ordered outcomes in which the sets of players strategies are finite. Then the set $Ac\, G$ of acceptable outcomes in game G is non-empty.*

A proof of corollary 3 can be reduced to previous corollary as follows. Let $G = \langle X, Y, A, \omega, F \rangle$ be an antagonistic game with ordered outcomes in which sets of players strategies X and Y are finite. Consider *the reduced game* G^0 which is obtained from the game G by elimination of non-realizable outcomes (that is in game G^0 the set of outcomes is the range of realization function F). Then in game G^0 the set of outcomes is finite and in accordance with corollary 2 the game has an acceptable outcome. It is easy to show that this outcome is also acceptable one in game G which was to be proved.

4.2. Uniqueness of acceptable outcome in a game having saddle point

Theorem 3. *Let $G = \langle X, Y, A, \omega, F \rangle$ be an antagonistic game with ordered outcomes which has a saddle point (x_0, y_0). Then the set $Ac\, G$ of acceptable points in game G consists of the outcome $F(x_0, y_0)$ and all outcomes which are non-comparable with element $F(x_0, y_0)$ under order ω.*

Proof (of theorem 3). Check that $F(x_0, y_0) \in U(2)$ (see (9)). Indeed, in accordance with the definition of saddle point, for any $x \in X$ there exists $y = y_0 \in Y$ such that $F(x, y) \overset{\omega}{\leq} F(x_0, y_0)$. Now assume that $a \in V^*(1)$. Then according to lemma 1 we obtain $a \overset{\omega}{<} F(x_0, y_0)$. Conversely, suppose $a \overset{\omega}{<} F(x_0, y_0)$. Then for any $y \in Y$ we get $F(x_0, y) \overset{\omega}{\geq} F(x_0, y_0) \overset{\omega}{>} a$ hence $F(x_0, y) \overset{\omega}{>} a$ i.e. $a \in V^*(1)$. Thus we show the formula:

$$V^*(1) = \{a \in A\colon a \overset{\omega}{<} F(x_0, y_0)\}. \tag{21}$$

The dual formula have the form:

$$V^*(2) = \{a \in A\colon a \overset{\omega}{>} F(x_0, y_0)\}. \tag{22}$$

Because the set $Ac\ G$ of acceptable points in game G can be presented as

$$Ac\ G = (V^*(1) \cup V^*(2))'$$

then using (21), (22) we obtain the proof of theorem 3. □

Corollary 4. *Let $G = \langle X, Y, A, \omega, F\rangle$ be an antagonistic game with ordered outcomes which has a saddle point (x_0, y_0). Then the set $Ac\ G$ of acceptable points in game G consists of a single element $F(x_0, y_0)$ if and only if any outcome of the game is comparable under order ω with outcome $F(x_0, y_0)$.*

4.3. Sufficient condition for uniqueness of acceptable outcome concerning the ordered set of outcomes

The following result was established in the paper (Rozen, 2018).

Theorem 4. *Let $G = \langle N, (X_i)_{i\in N}, A, (\rho_i)_{i\in N}, F\rangle$ be a game with quasi-ordered outcomes and for every $i \in N$ quasi-ordered set $\langle A, \rho_i\rangle$ ascending chain condition satisfies. Then there exists an unique up to the natural equivalence ε acceptable outcome if and only if game G has a special Nash equilibrium point x^0 and outcome $F(x^0)$ is a centric one.*

We now show that in the case of antagonistic game with ordered outcomes conditions of theorem 4 can be simplified by the following statements 1-4.

Statement 1. *In antagonistic game with ordered outcomes the concepts "special Nash equilibrium point" and "saddle point" are equivalent.*

Indeed, in antagonistic game with ordered outcomes Nash equilibrium point becomes a saddle point. It remains to show that any saddle point in antagonistic game is a special Nash equilibrium point. The last assertion means that for saddle point (x_0, y_0) the following condition

$$V^*(1) \cup V^*(2) = \{a \in A\colon a \overset{\omega}{<} F(x_0, y_0)\} \cup \{a \in A\colon a \overset{\omega}{>} F(x_0, y_0)\} \tag{23}$$

holds. The equality (23) follows directly from (21) and (22).

Statement 2. *In antagonistic game with ordered outcomes the natural equivalence relation ε is the identity relation Id_A:*

$$\varepsilon = \varepsilon_1 \cap \varepsilon_2 = \left(\omega \cap \omega^{-1}\right) \cap \left(\omega^{-1} \cap \omega\right) = Id_A \cap Id_A = Id_A. \tag{24}$$

Statement 3. *In antagonistic game with ordered outcomes, an element $c \in A$ is a centric one if and only if for any $a \in A$ the following condition holds*:

$$a = c \quad or \quad a \overset{\omega}{<} c \quad or \quad a \overset{\omega}{>} c. \tag{25}$$

The statement 3 follows from definitions and statement 2.

Statement 4. *In antagonistic game with ordered outcomes both ascending chain conditions (AC) for ordered sets $\langle A, \omega \rangle$ and $\langle A, \omega^{-1} \rangle$ are equivalent that all chains in $\langle A, \omega \rangle$ are finite (see Szasz, 1962).*

Using theorem 4 and statements 1–4, we now have the following result.

Theorem 5. *Let $G = \langle X, Y, A, \omega, F \rangle$ be an antagonistic game with ordered outcomes $\langle A, \omega \rangle$ in which all chains are finite. Then the set $Ac\ G$ of acceptable points in game G consists of a single element if and only if the game G has a saddle point and any outcome of the game is comparable under order ω with outcome at that saddle point.*

Remark 3. If the finiteness of chains condition in ordered set $\langle A, \omega \rangle$ is rejected, the statement of theorem 5 becomes false. Indeed, consider the following example.

Example 4. Let $G = \langle X, Y, F \rangle$ be antagonistic game with payoff function $F(x, y)$ given by the table 5 where a is a fixed real number. Recall that we can mean G as a game with ordered outcomes (see sec. 1.3).

Table 5. Payoff function F

$X \backslash Y$	y_1	y_2	y_3	$\cdots$	y_n	$\cdots$	inf
x_1	a	$a+1/2$	$a+1/3$	$\cdots$	$a+1/n$	$\cdots$	a
x_2	$a+1$	$a-1/2$	$a-1/3$	$\cdots$	$a-1/n$	$\cdots$	$a-1/2$
sup	$a+1$	$a+1/2$	$a+1/3$	$\cdots$	$a+1/n$	$\cdots$	$v_1 = v_2 = a$

We find $v_1 = \sup\limits_{x \in X} \inf\limits_{y \in Y} F(x, y) = a$, $v_2 = \inf\limits_{y \in Y} \sup\limits_{x \in X} F(x, y) = a$ i.e. game G has a value a in pure strategies. In this game the outcome a is an acceptable one for player 1 (since both strategies x_1 and x_2 are not objections of player 1 to outcome a). In addition, the outcome a is an acceptable one for player 2 (since no objections of player 2 to outcome a). Thus the outcome a is an acceptable one in game G. Moreover in game G no acceptable outcomes other than the outcome a. Indeed, the strategy $x_1 \in X$ is an objection of player 1 to any outcome $p < a$. Besides for any outcome $q > a$ there exists a strategy $y_n \in Y$ which is an objection of player 2 to outcome q (it is sufficiently put $n = [1/(q-a)] + 1$). Therefore in this game G the outcome a is the single acceptable one, however the game G has not of saddle points in pure strategies.

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