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Envy Stable Solutions for Allocation Problems with Public Resourses

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Abstract We consider problems of "fair" distribution of several different public resourses. If τ is a partition of a finite set N, each resourse c_j is distributed between points of $B_j \in \tau$. We suppose that either all resourses are goods or all resourses are bads. There are finite projects, each project use points from its subset of N (its coalition). $\mathcal{A}$ is the set of such coalitions. The gain/loss function of a project at an allocation depends only on the restriction of the allocation on the coalition of the project. The following 4 solutions are considered: the lexicographically maxmin solution, the lexicographically minmax solution, a generalization of Wardrop solution. For fixed collection of gain/loss functions, we define envy stable allocations with respect to Γ, where the projects compare their gains/losses at fixed allocation if their coalitions are adjacent in Γ. We describe conditions on $\mathcal{A}$, τ, and Γ that ensure the existence of envy stable solutions, and conditions that ensure the enclusion of the first three solutions in envy stable solution.

Keywords: lexicographically maxmin solution, Wardrop equilibrium, envy stable solution, equal sacrifice solution.

1. Introduction

We consider problems of fair allocation of several public resourses $c_j > 0$ between points of finite set in the case when either all resourses are public goods or all resourses are public bads. All resourses are distributed between points of finite set N as follows. For partition τ of N, a resourse c_j is distributed between points of $B_j \in \tau$. There are finite projects, for each project a its gain/loss depends on restriction of allocations on $S(a) \subset N$, i.e., $S(a)$ is a coalition of a. We suppose that $S(a)$ are different for different a and denote the set of such coalitions by $\mathcal{A}$.

The same problems arise when different B_j correspond to different moments or when different B_j correspond to different financial sourses.

We use the following notations.

For fixed τ, let $C = \{c_j = c(B_j)\}_{B_j \in \tau}$,

$$X = X(\tau, C) = \{x \in R^n : x_i \geq 0, \sum_{i \in B_j} x_i = c_j, B_j \in \tau\}$$

be the set of **imputations**. For $S \in \mathcal{A}$, $x \in X$, let $x_S = \{x_i\}_{i \in S}$,
G_S be a continuous strictly increasing in each variable function defined on $X_S = \{x_S : x \in X\}$, Γ be an undirected graph, where $\mathcal{A}$ is the set of nodes. We also denote $G_S(x) = G_S(x_S)$ for $x \in X$.

Special cases were considered for TU-cooperative games with restricted cooperation in (Naumova, 2011; Naumova, 2012). In that models $\tau = \{N\}$ and $G_S(x)$ was either excess or proportional excess of coalition S at imputation x. The case when $G_S(x_S) = U(x(S)) - U(v(S))$, was considered in (Naumova, 2013).

For the case when $G_S(x_S)$ are the gains of S at x, one of the natural solutions is the lexicographically maxmin solution (see Sudholter and Peleg, 1998 for example). It generalizes the nucleolus of cooperative game.

If $G_S(x_S)$ are the losses of S at x, then the lexicographically minmax solution seems to be natural. It generalizes the antinucleolus.

For continuous G_S, these solutions always exist. In fact, it was proved in (Schmeidler,1969) and formally proved in (Vilkov, 1974) for each compact set X. If, moreover, G_S are concave functions, then $G_S(x_S)$ are uniquely determined in lexicographically maxmin solution. If G_S are convex functions, then $G_S(x_S)$ are uniquely determined in lexicographically minmax solution.

In this paper we consider a new solution for these problems and call it envy stable solution. Let γ be an undirected graph, where $\mathcal{A}$ is the set of vertices. Projects compare their gains/losses iff their coalitions are adjacent in Γ. At allocation in envy stable solution, even if a project envies to another project, it can't object against the allocation. Formally, envy stable solution generalizes sequal sacrifice solution for claim problems, where all coalitions were singletons, each pair of singletons were adjacent in Γ and $\tau = \{N\}$.

All results of this paper concern characterisations of the collection of coalitions $\mathcal{A}$, the partition τ of N and the graph Γ that either ensure existence result for envy stable solution or ensure enclusion of other solutions in envy stable solution.

These conditions generalize the previous results of the author that were obtained in (Naumova, 2011; Naumova, 2012) only for two types of functions G_S (excesses and proportional excesses), $\tau = \{N\}$, and two types of graph Γ (either each two different coalitions are adjacent in Γ of all pairs of disjoin coalitions are adjacent in Γ.)

The problem of finding points in envy stable solution arises. We consider the possibilities to use other solutions as subsets of envy stable solution. The results are obtained only on special classes of functions G_S. For problems of enclusion lexicographically maxmin/minmax solution in envy stable solution, we take the set of functions $\mathcal{G}_S^\tau = \{G_S : G_S(x_S) = g_S(\{x(S \cap B)\}_{B \in \tau})\}$.

The problem of inclusion Wardrop equilibrium solution in envy stable one is considered for the case when $G_S(x) = g_S(x(S))$. Wardrop equilibrium is natural for road traffic problems, but we take it because it is the solution of a special minimization problem that can be solved by standart methods for some functions.

The paper is organized as follows. In Section 2. we define envy stable solution on $X(\tau, C)$ with respect to Γ and describe completely conditions on $\mathcal{A}$, τ, and Γ that ensure existence of $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution with respect to Γ for all continuous strictly increasing in each variable functions G_S.

In Section 3. we describe completely conditions on $\mathcal{A}$, τ, and Γ that ensure inclusion of lexicographically maxmin/minmax solutions and Wardrop equilibrium solution in $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution with respect to Γ.

2. Envy stable solution

In this section we define envy stable solutions and describe conditions that ensure its existence.

Definition 1. Let Γ be an undirected graph, where $\mathcal{A}$ is the set of nodes.

Let $\mathcal{A} \subset 2^N$, τ, Γ be fixed. An imputation $x \in X(\tau, C)$ belongs to $\{G_S\}_{S\in\mathcal{A}}$ - **envy stable solution w.r.t.** Γ if for each arc (P,Q) of Γ,
$G_P(x_P) > G_Q(x_Q)$ implies $x(P) = 0$.

This definition is a simplification of the definition of the generalized kernel for games with restricted cooperation and a generalization of equal sacrifice solution for claim problems.

It has the following motivation. For $x \in X$, only coalitions that are adjaent in Γ compare the values of their gains/losses. Let x belong to envy stable solution. If goods are distributed and Q envies to P at x then Q can't object against P since P does not use goods. If bads are distributed and P envy to Q at x, then P can't object against Q because P does not get any bads at x.

The following theorem describes existence condition of envy stable solution.

Theorem 1. *Let $\mathcal{A} \subset 2^N$, τ, Γ, C be fixed. For all continuous strictly increasing in each variable functions G_S ($S \in \mathcal{A}$), the $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ is a nonempty set if and only if $\mathcal{A}$, Γ, τ satisfy the following condition.*

C0($\mathcal{A}, \Gamma, \tau$). If a single node is taken out from each component of Γ, then each $B \in \tau$ is not covered by the remaining elements of $\mathcal{A}$.

Proof. Let $\mathcal{A}$, Γ, τ satisfy the condition C0$(\mathcal{A}, \Gamma, \tau)$, $\{G_S\}_{S\in\mathcal{A}}$ be a collection of continuous strictly increasing in each variable functions. For $x \in X(\tau, C)$ we define a directed binary relation $\succ_x$ on $\mathcal{A}$ as follows. $K \succ_x L$ iff K, L are adjacent in Γ, $G_K(x) < G_L(x)$ and $x(L) > 0$. Then $\succ_x$ is an acyclic binary relation. Let

$$F^L = \{x \in X(\tau, C) : K \not\succ_x L \quad \text{for all} \quad K \in \mathcal{A}\}.$$

Then F^L is a closed set because the functions G_S are continuous ones. In view of C0$(\mathcal{A}, \Gamma, \tau)$, it follows from Theorem 2 in Naumova, 1983 that there exists $x^0 \in X(\tau, C)$ such that $x^0 \in F^L$ for all $L \in \mathcal{A}$ because the collection of binary relations $\{\succ_x\}_{x\in X(\tau,C)}$ is an M-system of relations (in terms of Naumova, 1983).

Now suppose that the condition C0$(\mathcal{A}, \Gamma, \tau)$ is not fulfilled. Let $\{\mathcal{D}_k\}_{k=1}^m$ be a collection of all components of Γ. Then there exist $B^0 \in \tau$ and a collection $\{S_k\} \subset \mathcal{A}$ such that $S_k \in \mathcal{D}_k$ and for $\mathcal{A}^0 = \mathcal{A} \setminus \bigcup \{S_k\}_{k=1}^m$,

$$B^0 \subset \bigcup_{T\in\mathcal{A}^0} T.$$

Let $\bar{c} = \sum_{B\in\tau} c(B)$, $0 < \epsilon < c(B^0)/(\bar{c}|B^0|)$. Take the following collection of functions $\{G_S\}_{S\in\mathcal{A}}$.
$G_S(x_S) = x(S)$ for $S = S_k$, $k = 1, \dots m$
$G_S(x_S) = x(S)/\epsilon$ otherwise.
Let $y \in X(\tau, C)$ and y belong to $\{G_S\}_{S\in\mathcal{A}}$ - envy stable solution w.r.t. Γ. Then for $T \in \mathcal{A}^0$ we have

$$y(T) \le \epsilon\bar{c} < c(B^0)/|B^0|.$$

Thus, $y_i < c(B^0)/|B^0|$ at each $i \in B^0$, hence $y(B^0) < c(B^0)$ and $y \notin X(\tau, C)$. □

Example 1. Let $\mathcal{A}$ covers N, all nodes of Γ be adjacent. Then C0$(\mathcal{A}, \Gamma, \tau)$ is fulfilled iff $\mathcal{A}$ is a minimal covering of each $B \in \tau$.

Example 2. $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$. $|N| = 4$, $\tau = \{N\}$, $\mathcal{A}$ consists of no more than 5 two-person coalitions. Then $\mathrm{C0}(\mathcal{A}, \Gamma, \tau)$ is fulfilled.

Example 3. $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$. $|N| = 4$, $\tau = \{N\}$, $\mathcal{A}$ consists of all two-person coalitions. Then condition $\mathrm{C0}(\mathcal{A}, \Gamma, \tau)$ does not fulfill. Indeed, if we take off the coalitions $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, then the remaining elements of $\mathcal{A}$ cover N.

Example 4. $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$.
$N = \{1, \dots 6\}$, $\tau = \{\{1,2,3,4\}, \{5,6\}\}$,
$\mathcal{A} = \{\{1,2,5\}, \{3,4,6\}, \{1,3\}, \{2,4\}\}$. Then $\mathrm{C0}(\mathcal{A}, \Gamma, \tau)$ is fulfilled.

3. Selectors of envy-stable solution

In this section we define lexicographically maxmin solution, lexicographically minmax solution, Wardrop equilibrium solution and consider the problems of inclusion these solutions in envy stable solution. Here we restrict the class of gain/loss functions of coalitions as follows.

If τ is a partition of N, $S \in \mathcal{A}$, denote by $\mathcal{G}_S^\tau$ the set of G_S such that $G_S(x_S) = g_S(\{x(S \cap B)\}_{B \in \tau})$, where g_S are continuous strictly increasing in each variable functions.

3.1. Lexicographically maxmin solution

The following solution seems natural if $G_S(x_S)$ are gains of S at x.

Definition 2. For $x \in X(\tau, C)$, $\{G_S\}_{S \in \mathcal{A}}$, $k = |\mathcal{A}|$, we enumerate $S \in \mathcal{A}$, such that $G_{S_1}(x_{S_1}) \le G_{S_2}(x_{S_2}) \le \dots G_{S_k}(x_{S_k})$.
Denote $\theta(x) = (G_{S_1}(x_{S_1}), G_{S_2}(x_{S_2}), \dots G_{S_k}(x_{S_k}))$.

A vector $x \in X$ belongs to the $(\{G_S\}_{S \in \mathcal{A}}, \tau, C)$–**lexicographically maxmin solution** if $\theta(y) \not>_{lex} \theta(x)$ for all $y \in X(\tau, C)$.

If G_S are concave functions then for each $S \in \mathcal{A}$, $G_S(y_S)$ coincide at all y in this solution.

Lexicographically maxmin solution coincides with restricted nucleolus of cooperative game if $G_S(x_S) = G_S^1(x_S) = x(S) - v(S)$ and coincides with restricted proportional nucleolus if $G_S(x_S) = G_S^2(x_S) = x(S)/v(S)$.

For $i \in N$, denote $\mathcal{A}_i = \{T \in \mathcal{A} : i \in T\}$.

Definition 3. $\mathcal{A}$ is a (Γ, τ)–**positive mixed** collection of coalitions if for each $i \in B \in \tau$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$, and Q, S adjacent at Γ, there exists $j \in B$ such that $\mathcal{A}_j \supset \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

(For the case when $\tau = \{N\}$ and Q, S are adjacent in Γ iff $Q \cap S = \emptyset$, these collections are called weakly mixed in previous papers.)

Example 5. $|N| = 4$, $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$, $\tau = \{N\}$, $\mathcal{A} = \{\{i,j\},\ \{k,l\};\ \{i,k\},\ \{j,l\}\}$,
$\mathcal{A}$ is (Γ, τ)–positive mixed collection of coalitions.

Example 6. $|N| = 5$, $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$, $\tau = \{N\}$, $\mathcal{A} = \{\{i,j\},\ \{k,l,m\},\ \{i,k\},\ \{j,l\}\}$,
$\mathcal{A}$ is (Γ, τ)–positive mixed collection of coalitions.

Example 7. $|N| = 4$, $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$, $\tau = \{N\}$, $\mathcal{A} = \{\{i,j\},\ \{k\},\ \{i,m\}\}$,
$\mathcal{A}$ is not (Γ,τ)–positive mixed collection of coalitions. Indeed, take $Q = \{i,j\}$, $S = \{k\}$.

Theorem 2. *Let $S \cap Q = \emptyset$ for all $S, Q \in \mathcal{A}$ that are adjacent in Γ. For all $\{G_S\}_{S\in\mathcal{A}}$ with $G_S \in \mathcal{G}_S^\tau$, the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution is contained in the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ if and only if $\mathcal{A}$ is a (Γ,τ)–positive mixed collection of coalitions.*

Proof. Let $\mathcal{A}$ be a (Γ,τ)–positive mixed collection of coalitions and x belong to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution. Suppose that x does not belong to $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ, i.e., there exist $S, Q \in \mathcal{A}$ such that (S,Q) is an arc of Γ, $G_S(x_S) > G_Q(x_Q)$ and $x(S) > 0$. Take $i_0 \in S$ such that $x_{i_0} > 0$. Then $i_0 \notin Q$ as if $S \cap Q = \emptyset$. Let $i_0 \in B \in \tau$.

Since $\mathcal{A}$ is a (Γ,τ)–positive mixed collection of coalitions, there exists $j \in B$ such that $\mathcal{A}_j \supset \mathcal{A}_{i_0} \cup \{Q\} \setminus \{S\}$.

For $\delta > 0$, let $y^\delta = \{y_i\}_{i\in N}$, where $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_t = x_t$ otherwise. Take $\delta > 0$ such that $\delta < x_{i_0}$ and

$$G_Q(y_Q^\delta) < G_S(y_S^\delta).$$

Then $G_P(y_P^\delta) < G_P(x_P^\delta)$ only for $P = S$ and $G_Q(y_Q^\delta) > G_Q(x_Q)$. Since $G_Q(y_Q^\delta) < G_S(y_S^\delta)$, we obtain $\theta(y^\delta) >_{lex} \theta(x)$ and this contradicts the definition of the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution.

Now let the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution be always contained in the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ.

Suppose that $\mathcal{A}$ is not a (Γ,τ)–positive mixed collection of coalitions. Then there exist $B \in \tau$, $i_0 \in B$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ such that S and Q are adjacent in Γ, $S \cap Q = \emptyset$, and $\mathcal{A}_j \not\supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$ for all $j \in B$.

Let $0 < \epsilon < 1/(|\tau||N|)$.

We take allocation problem with $c_j = c_{B_j} = 1/|\tau|$ at each $B \in \tau$ and the following $\{G_S\}_{S\in\mathcal{A}}$:
$G_S(x_S) = x(S)$,
$G_P(x_P) = x(P)/|N|^2$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$G_T(x_T) = x(T)/\epsilon$ otherwise.

Let x belong to the $(\{G_S(x_S)\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution and to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ.

Since x belongs to the $(\{G_S(x_S)\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution, $x(P) > 0$ for each $P \in \mathcal{A}$. Then since x belongs to $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution, we have $x(Q) = \epsilon x(S) \leq \epsilon$.

There exists $j_0 \in B$ such that $x_{j_0} \geq 1/(|\tau||B|) \geq 1/(|\tau||N|)$. Then $j_0 \notin Q$ and $j_0 \neq i_0$. Hence, $|B| \geq 2$ and $|\tau| < |N|$.

Let $\delta > 0$, $y^\delta = \{y_i\}_{i\in N}$, where $y_{i_0} = x_{i_0} + \delta$, $y_{j_0} = x_{j_0} - \delta$, $y_i = x_i$ otherwise. We can take δ such that $\delta < 1/(|\tau||N|)$ and for each $T, P \in \mathcal{A}$,

$$G_T(x_T) < G_P(x_P) \quad \text{implies} \quad G_T(y_T^\delta) < G_P(y_P^\delta).$$

Then $y^\delta(P) > x(P)$ for each $P \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0}$. In particular, $y^\delta(Q) > x(Q)$, i.e., $G_Q(y_Q^\delta) > G_Q(x_Q)$.

It remains to prove that $\theta(y^\delta) >_{lex} \theta(x)$, i.e., x does not belong to the $\{G_S(x_S)\}_{S\in\mathcal{A}}, \tau, C\}$–lexicographically maxmin solution. In order to prove that, we shall prove the following.

There exists $P \in \mathcal{A}$ such that $G_P(y^\delta_P) > G_P(x_P)$ and $G_T(y^\delta_T) < G_T(x_T)$ implies $G_P(y^\delta_P) < G_T(y_T)$. By the choice of δ, it is sufficient to check that $G_P(x_P) < G_T(x_T)$.

Consider 2 cases.

Case 1. $j_0 \notin S$. Let $G_T(y^\delta_T) < G_T(x_T)$, then $T \ni j_0$ and $G_T(x_T) = x(T)/\epsilon$, hence $G_T(x_T) \geq x_{j_0}/\epsilon > 1$. Since $G_Q(x_Q) = x(Q)/\epsilon < 1$ and $y^\delta(Q) > x(Q)$, we can take $P = Q$.

Case 2. $j_0 \in S$, then there exists $P \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\}$. Then $G_P(x_P) = x(P)/(|N|^2) \leq 1/(|N|^2)$ and $G_P(y^\delta_P) > G_P(x_P)$. If $y^\delta(T) < x(T)$ then either $T = S$ and $G_S(x_S) = x(S) \geq 1/(|\tau||N|) > 1/(|N|^2)$ or $G_T(x_T) = x(T)/\epsilon$ and $G_T(x_T) \geq x_{j_0}/\epsilon > 1$. Thus, $y^\delta(T) < x(T)$ implies $G_T(x_T) > G_P(x_P)$. □

Remark 1. If some $S, Q \in \mathcal{A}$ with $S \cap Q \neq \emptyset$ are adjacent in Γ, then there exists an allocation problem, where $G_S(x_S) = k_S x(S)$ with $k_S > 0$ for $S \in \mathcal{A}$ such that the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution does not intersect the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ.

Proof. Let $\delta < 1/(6|N|)$. Consider the following allocation problem: $\tau = \{N\}$, $c_N = 1$,
$G_S(x_S) = x(S)$,
$G_Q(x_Q) = 2x(Q)$,
$G_T(x_T) = x(T)/\delta$ otherwise.
Suppose that x is contained in both sets. Then $x(S) > 0$ and $x(Q) > 0$ because x belongs to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution. Hence, $G_S(x_S) = G_Q(x_Q)$ and $x(Q) \leq 1/2$. Then there exists $j_0 \in N \setminus Q$ such that $x_{j_0} \geq 1/(2|N|) > 3\delta$. Let $i_0 \in S \cap Q$.

Take the following y:
$y_{j_0} = x_{j_0} - \delta$,
$y_{i_0} = x_{i_0} + \delta$,
$y_i = x_i$ otherwise.
Then $y(S) \geq x(S)$, $y(Q) > x(Q)$ and $y(T) < x(T)$ implies $y(T) > 2\delta$, i.e., $G_T(y_T) > 2$. Since $G_S(y_S) \leq 1$, $G_Q(y_Q) \leq 2$, we obtain $G_T(y_T) > G_Q(y_Q)$, $G_T(y_T) > G_S(y_S)$, so $\theta(y) >_{lex} \theta(x)$ and x does not belong to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution. □

Remark 2. Theorem 2 is not valid without the condition $G_S \in \mathcal{G}^\tau_S$. We demonstrate this by the following example.

Example 8. $N = \{1, 2, 3, 4\}$, $\tau = \{N\}$, $c = c_N = 2$,
$\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$,
the arcs of Γ are $(\{1, 2\}, \{3, 4\})$ and $(\{1, 3\}, \{2, 4\})$,
$G_{\{1,2\}}(x) = x_1 + x_2$, $G_{\{3,4\}}(x) = x_3 + x_4$, $G_{\{1,3\}}(x) = x_1/2 + x_3$, $G_{\{2,4\}}(x) = x_2/2 + x_4$.

Let x belong to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution and to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ. Since x belongs to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ, the cases $x(\{1, 2\}) = 0$, $x(\{3, 4\}) = 0$, $x(\{1, 3\}) = 0$ and

$x(\{2,4\}) = 0$ are impossible, so we have $x_1 + x_2 = x_3 + x_4 = 1$, $G_{\{1,3\}}(x) = G_{\{2,4\}}(x) = 1 - x(\{1,2\})/4 = 3/4$ and $\theta(x) = (3/4, 3/4, 1, 1)$. But for $y = (2/5, 2/5, 3/5, 3/5)$, $\theta(y) = (4/5, 4/5, 4/5, 6/5)$ and $\theta(y) >_{lex} \theta(x)$ and x does not belong to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically maxmin solution.

3.2. Lexicographically minmax solution

If the functions $G_S(x_S)$ are losses of S at x, it is natural to consider the folowing solution.

Definition 4. For $x \in X(\tau, C)$, $\{G_S\}_{S\in\mathcal{A}}$, $k = |\mathcal{A}|$, we enumerate $S \in \mathcal{A}$, such that $G_{S_1}(x_{S_1}) \geq G_{S_2}(x_{S_2}) \geq \dots G_{S_k}(x_{S_k})$.
Denote $\bar{\theta}(x) = (G_{S_1}(x_{S_1}), G_{S_2}(x_{S_2}), \dots G_{S_k}(x_{S_k}))$.

A vector $x \in X$ belongs to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–**lexicographically minmax solution** if $\bar{\theta}(y) \not<_{lex} \bar{\theta}(x)$ for all $y \in X(\tau, C)$.

If G_S are convex functions then for each $S \in \mathcal{A}$, $G_S(y_S)$ coincide at all y in this solution.

Lexicographically minmax solution coincides with restricted antinucleolus of cooperative game if $G_S(x_S) = G^1_S(x_S) = x(S) - v(S)$ and coincides with restricted proportional antinucleolus if $G_S(x_S) = G^2_S(x_S) = x(S)/v(S)$.

Definition 5. $\mathcal{A}$ is a (Γ, τ) **– negative mixed** collection of coalitions if for each $i \in B \in \tau$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$, and Q, S adjacent at Γ, there exists $j \in B$ such that $\mathcal{A}_j \subset \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Example 9. Let $\tau = \{N\}$. If $\mathcal{A}$ is a minimal cover of N, then $\mathcal{A}$ is a Γ, τ – negative mixed collection of coalitions for each Γ.

Example 10. $|N| = 5$, $K, L \in \mathcal{A}$ are adjacent in Γ iff $K \cap L = \emptyset$, $\tau = \{N\}$,
$\mathcal{A} = \{\{1,2\}, \{3,4,5\}, \{1,3\}, \{2,4\}\}$,
$\mathcal{A}$ is not a (Γ, τ)–negative mixed collection of coalitions. Indeed, take $Q = \{3,4,5\}$, $i = 5$, $S = \{1,2\}$.

Theorem 3. *For all $\{G_S\}_{S\in\mathcal{A}}$ with $G_S \in \mathcal{G}^\tau_S$, the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically minmax solution is contained in the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ if and only if $\mathcal{A}$ is a (Γ, τ)–negative mixed collection of coalitions.*

Proof. Let $\mathcal{A}$ be a (Γ, τ)–negative mixed collection of coalitions and x belong to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically minmax solution. Suppose that x does not belong to $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ, i.e., there exist $S, Q \in \mathcal{A}$ such that (S, Q) is an arc of Γ, $G_S(x_S) > G_Q(x_Q)$ and $x(S) > 0$. Take $i_0 \in S$ such that $x_{i_0} > 0$. Let $i_0 \in B \in \tau$.

Since $\mathcal{A}$ is a (Γ, τ)–negative mixed collection of coalitions, there exists $j \in B$ such that $\mathcal{A}_j \subset \mathcal{A}_{i_0} \cup \{Q\} \setminus \{S\}$. Then $j \notin S$.

For $\delta > 0$, let $y^\delta = \{y_i\}_{i\in N}$, where $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_t = x_t$ otherwise. Take $\delta > 0$ such that $\delta < x_{i_0}$ and $G_Q(y^\delta_Q) < G_S(y^\delta_S)$.

Then $G_S(y^\delta < G_S(x)$ because $j \notin S$ and $G_P(y^\delta) > G_P(x)$ only for $P = Q$. Thus, $\bar{\theta}(y^\delta) <_{lex} \bar{\theta}(x)$ and this contradicts the definition of the $(G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically minmax solution.

Now let the $(\{G_S\}_{S\in\mathcal{A}},\tau,C)$–lexicographically minmax solution be always contained in the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ.

Suppose that $\mathcal{A}$ is not a (Γ,τ)–negative mixed collection of coalitions. Then there exist $B\in\tau$, $i_0\in B$, $Q\in\mathcal{A}_{i_0}$, and $S\in\mathcal{A}$ such that S and Q are adjacent in Γ, and $\mathcal{A}_j\not\subset\mathcal{A}_{i_0}\cup\{S\}\setminus\{Q\}$ for all $j\in B$.

Let $0<\epsilon<1/(|\tau||N|)$, $M>1$.

We take the allocation problem with $c_j=c_{B_j}=1/|\tau|$ at each $B\in\tau$ and the following $\{G_S\}_{S\in\mathcal{A}}$:
$G_S(x_S)=x(S)$,
$G_P(x_P)=x(P)/M$ for $P\in\mathcal{A}_{i_0}\setminus\{Q\}$,
$G_T(x_T)=x(T)/\epsilon$ otherwise.

Let x belong to the $(\{G_S(x_S)\}_{S\in\mathcal{A}},\tau,C)$–lexicographically minmax solution and to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ.

If $x(Q)>\epsilon$ then $x(Q)/\epsilon\le x(S)\le 1$ because x belongs to the $(\{G_S(x_S)\}_{S\in\mathcal{A}},\tau,$ $C)$–lexicographically minmax solution, hence $x(Q)\le\epsilon$. for each $P\in\mathcal{A}$. Then since x belongs to $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution, we have $x(Q)=\epsilon x(S)\le\epsilon$.

There exists $j_0\in B$ such that $x_{j_0}\ge 1/(|\tau||B|)\ge 1/(|\tau||N|)>\epsilon$. Then $j_0\notin Q$ and $j_0\ne i_0$.

Let $\delta>0$, $y^\delta=\{y_i\}_{i\in N}$, where $y_{i_0}=x_{i_0}+\delta$, $y_{j_0}=x_{j_0}-\delta$, $y_i=x_i$ otherwise. We can take δ such that $\delta<1/(|\tau||N|)$ and for each $T,P\in\mathcal{A}$,

$$G_T(x_T)<G_P(x_P)\quad\text{implies}\quad G_T(y^\delta_T)<G_P(y^\delta_P).$$

Since $\mathcal{A}_j\not\subset\mathcal{A}_{i_0}\cup\{S\}\setminus\{Q\}$ for all $j\in B$, there exists $\bar P\in\mathcal{A}$ such that $j_0\in\bar P$ and $\bar P\notin\mathcal{A}_{i_0}\cup\{S\}\setminus\{Q\}$. Then $\bar P\ne Q$ as if $j_0\notin Q$, hence $i_0\notin\bar P$. This implies $y^\delta(\bar P)<x(\bar P)$.

Moreover, $\bar P\ne S$ implies $G_{\bar P}(x)=x(\bar P)/\epsilon$, hence

$$G_{\bar P}(x)\ge x_{j_0}/\epsilon>1.$$

Let $y^\delta(T)>x(T)$ then $i_0\in T$, $j_0\notin T$. Then either $T=Q$ and $G_T(x)=x(Q)/\epsilon\ge 1$ or $G_T(x)=x(T)/M<1$, thus,

$$G_{\bar P}(x)>G_T(x).$$

It follows from the choice of δ that $G_{\bar P}(y^\delta)>G_T(y^\delta)$ for each T with $y^\delta(T)>x(T)$.

Thus, $\bar\theta(y^\delta)<_{lex}\bar\theta(x)$ and this contradicts the definition of the $(G_S\}_{S\in\mathcal{A}},\tau,C)$–lexicographically minmax solution. □

Chernyshova proved this theorem for the case when $\tau=\{N\}$, the coalitions are adjacent in Γ iff they are disjoint, and $G_S(x)=x(S)/v(S)$ (see Chernysheva, 2017).

Remark 3. Theorem is not valid without the condition $G_S\in\mathcal{G}^\tau_S$. We demonstrate this by the following example.

Example 11. $N=\{1,2,3,4\}$, $\tau=\{N\}$, $c=c_N=2$,
$\mathcal{A}=\{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$,
the arcs of Γ are $(\{1,2\},\{3,4\})$ and $(\{1,3\},\{2,4\})$,
$G_{\{1,2\}}(x)=x_1+x_2$, $G_{\{3,4\}}(x)=x_3+x_4$, $G_{\{1,3\}}(x)=2x_1+x_3$, $G_{\{2,4\}}(x)=2x_2+x_4$.

Let x belong to the $(\{G_S\}_{S\in\mathcal{A}},\tau,C)$–lexicographically minmax solution and to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ. Since x belongs to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ, we have $x_1 + x_2 = x_3 + x_4 = 1$, $G_{\{1,3\}}(x) = G_{\{2,4\}}(x) = 1 + x(\{1,2\})/2 = 3/2$ and $\bar{\theta}(x) = (3/2, 3/2, 1, 1)$. But for $y = (1/3, 1/3, 2/3, 2/3)$, $\bar{\theta}(y) = (4/3, 4/3, 4/3, 2/3)$ and $\theta(y) <_{lex} \theta(x)$, and x does not belong to the $(\{G_S\}_{S\in\mathcal{A}}, \tau, C)$–lexicographically minmax solution.

4. Wardrop equilibrium

The following solution was defined in (Wardrop, 1952) for road traffic problems. In our model, it can be used if $G_S(x) = g_S(x(S))$, where g_S are functions defined on $[0, +\infty)$.

Definition 6. Let $\{g_S\}_{S\in\mathcal{A}}$ be a collection of strictly increasing continuous functions defined on $[0, +\infty)$, τ be a partition of N, $C = \{c_(B)\}_{B\in\tau}$. An allocation $x \in X(\tau, C)$ is a **Wardrop equilibrium with respect to** $\{g_S\}_{S\in\mathcal{A}}$ if for each $B \in \tau$, $i, j \in B$,

$$\sum_{T\in\mathcal{A}:\, T\ni i} g_T(x(T)) > \sum_{T\in\mathcal{A}:\, T\ni j} g_T(x(T)) \quad \text{implies} \quad x_i = 0.$$

Definition 7. Let $\{g_S\}_{S\in\mathcal{A}}$ be a collection of strictly increasing continuous functions defined on $[0, +\infty)$. A $(\{g_S\}_{S\in\mathcal{A}}, \tau, C)$–**solution** is the set of solutions of the problem
$\sum\limits_{S\in\mathcal{A}} \int\limits_0^{z(S)} g_S(t)dt \to min_{\{z:\ z\in X(\tau,C)\}}$.

The following fact is well known (see, for example, Krylatov and Zakharov, 2017 Th.1 or Mazalov, 2010 Th.9.10).

Proposition 1. *An allocation $x \in X(\tau, C)$ belongs to $(\{g_S\}_{S\in\mathcal{A}}, \tau, C)$–solution iff it is a Wardrop equilibrium with respect to $\{g_S\}_{S\in\mathcal{A}}$.*

Definition 8. A collection of coalitions $\mathcal{A}$ is a (Γ, τ)–**mixed collection** if for each $B \in \tau$, $i \in B$, $Q \in \mathcal{A}_i$ $S \in \mathcal{A}$ and Q, S adjacent at Γ, there exists $j \in B$ such that $\mathcal{A}_j = \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Example 12. $|N| = 4$, $\tau = \{N\}$, P and Q are adjacent in Γ iff $P \cap Q = \emptyset$, $\mathcal{A} = \{\{i, j\},\ \{k, l\};\ \{i, k\},\ \{j, l\}\}$, then $\mathcal{A}$ is a (Γ, τ)–mixed collection.

Example 13. $|N| = 5$, $\tau = \{N\}$, P and Q are adjacent in Γ iff $P \cap Q = \emptyset$, $\mathcal{A} = \{\{1, 2\},\ \{3, 4, 5\},\ \{1, 3\},\ \{2, 4\}\}$.
It was demonstrated in 10 that $\mathcal{A}$ is not a (Γ, τ)–negative mixedcollection, hence it is not a (Γ, τ)–mixed collection of coalitions.

Theorem 4. *For all collections $\{g_S\}_{S\in\mathcal{A}}$ of strictly increasing continuous functions defined on $[0, +\infty)$, if $G_S(x) = g_S(x(S))$, then the $(\{g_S\}_{S\in\mathcal{A}}, \tau, C)$–solutions are contained in the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ if and only if $\mathcal{A}$ is a (Γ, τ)–mixed collection of coalitions.*

Proof. Let $\mathcal{A}$ be a (Γ, τ)–mixed collection of coalitions. Let x belong to a $(\{g_S\}_{S\in\mathcal{A}}, \tau, C)$–solution.

Suppose that x does not belong to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ, i.e., there exist $Q, S \in \mathcal{A}$ such that (Q, S) is an arc of Γ, $x(Q) > 0$, and $g_Q(x(Q)) >$

$g_S(x(S))$. Take $i_0 \in Q$ with $x_{i_0} > 0$. Let $i_0 \in B \in \tau$. Since $\mathcal{A}$ is a (Γ, τ)–mixed collection, there exists $j_0 \in B$ such that $\mathcal{A}_{j_0} = \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Thus, $\{T \in \mathcal{A} : T \not\ni j_0, T \ni i_0\} = \{Q\}$, $\{T \in \mathcal{A} : T \ni j_0, T \not\ni i_0\} = \{S\}$, hence

$$\sum_{T\in\mathcal{A}:\, T\ni i_0} g_T(x(T)) - \sum_{T\in\mathcal{A}:\, T\ni j_0} g_T(x(T)) = g_Q(x(Q)) - g_S(x(S) > 0,$$

but this contradicts Proposition 1. Thus, x belongs to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ.

Let $(\{g_S\}_{S\in\mathcal{A}}, \tau, C)$–solution be always contained in the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ for $G_S(x) = g_S(x(S))$. Suppose that $\mathcal{A}$ is not a (Γ, τ)–mixed collection of coalitions. Then there exist $B \in \tau$, $i_0 \in B$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ such that (S, Q) is an arc of Γ, and for each $j \in B$, $\mathcal{A}_j \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$.

Let $0 < \epsilon < 1/(|\tau||N|)$, $M > 1$. We take the following problem. $c_B = 1/\tau$ for each $B \in \tau$,
$g_S(x(S)) = x(S) - 1$,
$g_P(x(P)) = x(P)/M - 1$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$g_T(x(T)) = x(T)/\epsilon - 1$ otherwise.

Let x belong to the $(\{g_S\}_{S\in\mathcal{A}}, \tau, C)$–solution and to the $\{G_S\}_{S\in\mathcal{A}}$–envy stable solution w.r.t. Γ. First, we prove that $x(Q) \leq \epsilon$. If $x(Q) > \epsilon$, then $g_Q(x(Q)) \leq g_S(x(S)$, i.e., $x(Q) \leq \epsilon x(S) \leq \epsilon$. There exists $j_0 \in B$ such that $x_{j_0} \geq 1/(|\tau||N|)$. Then $j_0 \notin Q$.

Note that $g_Q(x(Q)) \leq g_S(x(S))$ because if $x(Q) = 0$ then $g_Q(x(Q)) = -1$, $g_S(x(S)) > -1$ since $x_{j_0} > 0$, and if $x(Q) > 0$ then it follows from the definition of envy stable solution.

We shall prove that

$$\sum_{T\in\mathcal{A}:T\ni i_0} g(x(T)/v(T)) < \sum_{T\in\mathcal{A}:T\ni j_0} g(x(T)/v(T)), \tag{1}$$

and this will contradict Proposition 1.

Since $\mathcal{A}_{j_0} \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$, the following 3 cases are possible.
1. $j_0 \notin S$.
2. $j_0 \in S$, $\mathcal{A}_{i_0} \setminus \{Q\} \neq \emptyset$, and $j_0 \notin \bigcap_{P\in\mathcal{A}_{i_0}\setminus\{Q\}} P$.
3. $j_0 \in \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$ and there exists $T_0 \in \mathcal{A}_{j_0} \setminus (\mathcal{A}_{i_0} \cup \{S\})$.

Case 1. If $T \in \mathcal{A}_{i_0} \setminus \{Q\}$ then $g(T) < 0$. Moreover, $x(Q) < \epsilon$ in this case because $x(S) < 1$ and $x(Q) \geq \epsilon$ implies $x(Q) \leq \epsilon x(S) < \epsilon$. Thus, $g_Q(x(Q)) < 0$ and

$$\sum_{T\in\mathcal{A}:\, T\not\ni j_0, T\ni i_0} g_T((x(T)) \leq g_Q((x(Q)) < 0.$$

For all $T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$, $x(T) > \epsilon$ and $g_T(x(T)) > 0$ as if $j_0 \notin S$, therefore,

$$\sum_{T\in\mathcal{A}:\, T\ni j_0, T\not\ni i_0} g_T((x(T)) \geq 0,$$

this implies (10).

Case 2. Since $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\} \neq \emptyset$ and $g_T(x(T)) < 0$ for all $T \in \mathcal{A}_{i_0} \setminus \{Q\}$, we obtain

$$\sum_{T\in\mathcal{A}:\, T\not\ni j_0, T\ni i_0} g_T((x(T)) = g_Q((x(Q)) + \sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_0}\setminus\{Q\}} g_T((x(T)) < g_Q((x(Q).$$

If $T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$ then either $T = S$ or $x(T) > \epsilon$ and $g_T(x(T)) > 0$, therefore

$$\sum_{T \in \mathcal{A}:\, T \ni j_0, T \not\ni i_0} g_T((x(T)) \geq g_S((x(S)).$$

Since $g_Q(x(Q)) \leq g_S(x(S))$, we obtain (10).

Case 3. Here $\{T \in \mathcal{A} :\, T \not\ni j_0, T \ni i_0\} = \{Q\}$, so

$$\sum_{T \in \mathcal{A}:\, T \not\ni j_0, T \ni i_0} g_T((x(T)) = g_Q((x(Q)).$$

If $T \in \mathcal{A}_{j_0} \setminus (\mathcal{A}_{i_0} \cup \{S\})$ then $g_T(x(T)) \geq 0$, so

$$\sum_{T \in \mathcal{A}:\, T \ni j_0, T \not\ni i_0} g_T((x(T)) \geq g_S((x(S)) + g_{T_0}((x(T_0)) > g_S((x(S)).$$

Since $g_Q(x(Q)) \leq g_S(x(S))$, we obtain (1). □

5. Conclusion

The paper considered some solutions of allocation problems with different public resourses. Each coalition from a fixed collection of coalitions estimates an allocation by its gain/loss function, and the result of estimation depends only on restriction of allocation on that coalition. A new solution concept (envy stable solution) was introduced. Conditions on the collection of coalitions that ensure existence result at all continuous gain/loss functions of coalitions are described. The conditions on the collection of coalitions that ensure inclusion of lexmaxmin, lexminmax, and Wardrop equlibrium solutions in envy stable solution are described in terms of the collection of coalitions.

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