Contributions to Game Theory and Management, XII, 204–219

A New Allocation Rule for Cooperative Games with Hypergraph Communication Structure*

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Abstract A new allocation rule was proposed by splitting the original game into a game between hyperlinks and games within them. A special case of cooperative game on flower type hypergraphs is investigated. The proposed allocation rule has been generalized for games with a communication structure represented by acyclic reduced hypergraphs and was illustrated by examples.

Keywords: cooperation, characteristic function, hypergraph, communication structure, allocation rule.

1. Introduction

In a classical way for group $N := 1, ..., n$ of agents the economic possibilities of each subgroup are described by cooperative game (N, v), where N is a set of players and v is a characteristic function. The characteristic function shows the power of each coalition. In this paper, we assume the cooperative game with transferable utility or TU-games.

Classically in this game, we assume that each subset of players can decide to cooperate and the total payoff of this cooperation can be distributed among the players. But in many practical situations, not all players can communicate with each other due to some economic, technological or other reasons, thus some coalitions cannot be created. It is the class of TU-games with limited cooperation. The communication structure can be introduced by an undirected graph. In this way, just players who have a link between them can cooperate. These games were first studied in (Myerson, 1977), he introduced games on a graph and characterized the Shapley value (Shapley, 1953). Hereafter, games with communication structure have received a lot of attention in cooperative game theory. In (Owen, 1986) were studied games where the communication structure is a tree. The position value for games where communication structure is given by a graph was introduced in (Meessen, 1988).

But generally, the communication structure can be given by a graph or hypergraph. For example, it can be some companies or sports teams. Cooperation between two organizations is only possible if they have at least one member in both of them.

The TU-games on hypergraph were studied in (Nouweland, Borm and Tijs, 1992), they characterized the Myerson value and the position value for these games. The third value, which is called degree value for the games with hypergraph communication structure was introduced in (Shan, Zhang and Shan, 2018). Many allocation rules for TU-games with a hypergraph communication structure can be proposed

* This research was supported by Russian Science Foundation under grants No.17-11-01079.

based on some different interpretations. The Myerson value highlighting the role of the players, the position value focuses on the role of communication.

In this paper, we introduced a new allocation rule for TU-games on the hypergraph.

2. Game on subclass of hypergraph

2.1. Preliminaries

In this section, we recall some notations and definitions about TU-games and hypergraph.

TU-game is a pair (N, v). Characteristic function $v : 2^N \to \mathbb{R}$ and $v(\emptyset) = 0$. We will use $|S|$ to show the cardinality of any $S \in N$.

Hypergraph is a pair $(N, \mathscr{H})$, $\mathscr{H} \subseteq \{H \in 2^N \big| |H| \geqslant 2\}$. $\mathscr{H}$ is some set of subsets of players N with cardinality more or equal two.

2.2. Definition of the game

Let $N = \{1, \ldots, n-1, c\}$ be a player set. Communication possibilities described by hypergraph $(N, \mathscr{H})$ In this part we will consider a special communication structure which given by

$$\overline{\mathscr{H}} \subseteq \{H \in 2^N \big| |H| \geqslant 2,\ H_j \cap H_k = c;\ j \neq k\ \forall H_j, H_k \in \mathscr{H}\}.$$

The interpretation of this structure is there is just one player who included in all hyperlinks and other players included just in one of them. The communication is only possible between the players in hyperlinks. It can be also interpreted as the central player has some companies with workers.An example of this hypergraph shown in fig.1

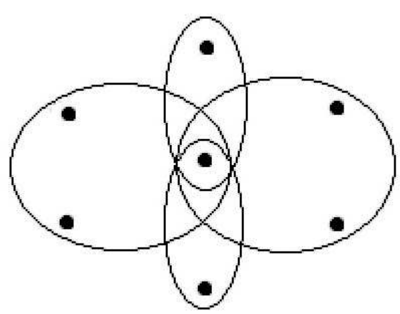

Fig. 1. An example of this hypergraph.

Denote the numbers of hyperlink in communication structure by L. Also denote the central-player by c. Let Γ_i be a set of players which included in hyperlink H_i except player c and U_i - the set of their strategies. Also denote a strategy of simple-player j as u^j. A strategy of player c from his set of strategies we will denote by $u_c \in \mathfrak{U}_c$. We define the payoff function of simple-player j in hyperlink H_i in this way

$$h_j(U_i, u_c) = K_j(U_i, u_c)$$

where K_j — payoff of player j which is defined on hyperlink which include player j. The payoff function of central-player c:

$$h_c(U_1, U_2, \ldots, U_L, u_c) = K_c^1(U_1, u_c) + K_c^2(U_2, u_c) + \cdots + K_c^L(U_L, u_c)$$

2.3. Cooperation

Now consider the case when the players agree to cooperate. It means that they will choose their strategies to maximize the sum of their payoffs

$$\sum_{k \in N} h_k = \sum_{i=1}^{L} \sum_{m \in H_i} K_m\left(U_i, u_c\right) + \sum_{j=1}^{L} K_c^j(U_j, u_c).$$

We suppose transferable payoffs. Thus the main question is how to allocate the total payoff between players. We will do it in three steps. On the first step, we construct a new cooperative game where we consider hyperlinks as players. We will create a characteristic function for all coalitions in this game. After that we solve this game proposing some allocation rule, in this paper we use a solution with equal excess. So we get the payoffs for all hyperlinks. The second step is to allocate this payoff between the members in a hyperlink. To solve this problem we will use the proportional solution. The last step is to find the total payoff for the central player. It will be the sum of his payoffs from all hyperlinks.

First step For now we consider the game where the players are hyperlinks from the given communication structure.The set of hyperlinks we will denote as $\mathcal{H}$. $\mathcal{S} \subseteq \mathcal{H}$ is coalition from this set of hyperlinks. We define the characteristic function as follows:

$$V(\mathcal{S}) = \sum_{i\colon\, H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K_j(\widetilde{U}_i, \hat{u}_c) + \sum_{i\colon\, H_i \in \mathcal{S}} K_c^i(\widetilde{U}_i, \hat{u}_c),$$

where $\hat{u}_c$ the solution of this maximization problem:

$$\max_{u_c} \max_{U_i} \left(\sum_{i\colon\, H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K_j(U_i, u_c) + \sum_{i\colon\, H_i \notin \mathcal{S}} K_c^i(U_i, u_c) \right) =$$

$$= \sum_{i\colon\, H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K_j(\widehat{U}_i, \widehat{u}_c) + \sum_{i\colon\, H_i \notin \mathcal{S}} K_c^i(\widehat{U}_i, \widehat{u}_c),$$

and $\widetilde{U}_i$ the solution of:

$$\max_{U_i} \left(\sum_{i\colon\, H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K_j(U_i, \widehat{u}_c) + \sum_{i\colon\, H_i \in \mathcal{S}} K_c^i(U_i, \widehat{u}_c) \right) =$$

$$= \sum_{i\colon\, H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K_j(\widetilde{\widetilde{U_i}}, \widehat{u}_c) + \sum_{i\colon\, H_i \in \mathcal{S}} K_c^i(\widetilde{U}_i, \widehat{u}_c)$$

$$V(\mathcal{H}) = \max_{u_c} \max_{U_i} \left(\sum_{i\colon\, H_i \in \mathcal{H}} \sum_{j \in \Gamma_i} K_j(U_i, u_c) + \sum_{i\colon\, H_i \in \mathcal{H}} K_c^i(U_i, u_c) \right)$$

We can interpret the values of characteristic in a following way. We suppose that central player is maximizing the total payoff of players in hyperlinks which are not in S. Based on this, the central playerвЂ™s $\widehat{u}_c$ strategy is chosen, assuming that in the worst case the central player will play this strategy, players from S seek to maximize their total payoff. Thus, we have determined the characteristic function for all coalitions of the hyperlinks. The next step is to find the gains for each hyperlink. For this, we use a solution with equal excess.

$$\xi_{H_j} = V(H_J) + \frac{V(\mathcal{H}) - \sum\limits_{i \in \mathcal{H}} V(H_i)}{L}, \quad j = \overline{1, L}.$$

Second step After the previous step the payoffs for each hyperlink have been received, the next step will be the distribution of this payoff between the players in each hyperlink. At this step, we get L cooperative games, for each we define the characteristic function for coalitions of players and the optimality principle. The characteristic function in these games will be determined in accordance with the approach described in (Neumann and Morgenstern, 1994). The idea is quite simple: the characteristic function in this case shows the maximum gain that a coalition can receive, provided that all other players play against it. As an optimality principle, we take a proportional solution. Consider a game on the hyperlink H_j. We denote the set of players on hyperlink H_j, including the central one, by N_j. We assume that $v^j(N_j) = \xi_{H_j}$. Define the value of the characteristic function for each of the simple players on this hyperlink H_j as

$$v^j(i) = \max_{u^i} \min_{u_c \bigcup U_j \setminus u^i} K_i(U_j, u_c).$$

for central-player on the same hyperlink

$$v^j(c) = \max_{u_c} \min_{U_j} K_c^j(U_j, u_c).$$

Now we can define the payoff of each simple-player on the hyperlink H_j as:

$$\mathcal{E}_i^j = \frac{v^j(i)}{\sum\limits_{k \in N_j} v^j(k)} v(N_j) = \frac{v^j(i)}{\sum\limits_{k \in N_j} v^j(k)} \xi_{H_i}.$$

The payoff of the central-player on the hyperlink H_j we will define by

$$\mathcal{E}_c^j = \frac{v^j(c)}{\sum\limits_{k \in N_j} v^j(k)} v(N_j) = \frac{v^j(c)}{\sum\limits_{k \in N_j} v^j(k)} \xi_{H_i}.$$

Third step We already defined the payoffs of all simple-players. The payoff of the central-player is said to be equal a sum of his payoffs on each hyperlink.

$$\mathcal{E}_c = \sum_{j=1}^{L} \mathcal{E}_c^j.$$

This is the main idea of this work.

2.4. Example

For better understanding we will use this solution on the example. Consider the cooperative game with player set $N = \{1,2,3,4,c\}$ and hypergraph $H_1 = \{1,2,c\}, H_2 = \{3,4,c\}$ which is shown on fig.2 .

For this example we consider that in each hyperlink players have bimatrix game between each other. It means that for simple-player j in hyperlink H_i payoff function is

$$h_j(U_i, u_c) = K_j(U_i, u_c) = \sum_{k: u^k \in U_i \setminus u^j} K_j(u^k, u^j) + K_j(u^j, u_c),$$

and for central player the payoff function

$$h_c(U_1, U_2, \dots, U_L, u_c) = K_c^1(U_1, u_c) + K_c^2(U_2, u_c) + \dots + K_c^L(U_L, u_c)$$

where

$$K_c^i(U_1, u_c) = \sum_{k: u^k \in U_i} K_c^i(u^k, u_c)$$

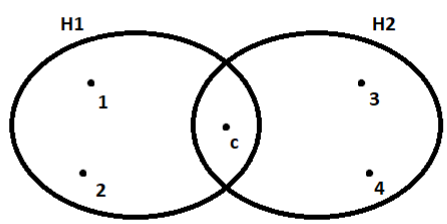

Fig. 2. Communication structure

Define the bimatrix game for each pair of linked players. We write a bimatrix 2×2 for player i and j where i chooses the row and j chooses column. We consider that all players have the set of strategies (A, B).

For players 1 and c

$$\begin{pmatrix} 4\backslash 8 & 3\backslash 6 \\ 1\backslash 3 & 5\backslash 6 \end{pmatrix}$$

For players 2 and c

$$\begin{pmatrix} 3\backslash 6 & 5\backslash 5 \\ 0\backslash 2 & 4\backslash 8 \end{pmatrix}$$

For players 1 and 2

$$\begin{pmatrix} 6\backslash 8 & 6\backslash 0 \\ 4\backslash 3 & 0\backslash 6 \end{pmatrix}$$

For players 3 and c

$$\begin{pmatrix} 8\backslash 0 & 6\backslash 10 \\ 3\backslash 6 & 9\backslash 3 \end{pmatrix}$$

For players 4 and c

$$\begin{pmatrix} 5\backslash 2 & 8\backslash 9 \\ 7\backslash 2 & 6\backslash 5 \end{pmatrix}$$

For players 3 and 4

$$\begin{pmatrix} 0\backslash 1 & 10\backslash 4 \\ 7\backslash 0 & 3\backslash 8 \end{pmatrix}$$

First step. Firstly we find the value of characteristic function for all coalitions of hyperlinks.

$$V(H_1) = \sum_{j \in \Gamma_1} K_j(\widetilde{U}_1, \hat{u}_c) + K_c^1(\widetilde{U}_1, \hat{u}_c),$$

where $\hat{u}_c$ the solution of this maximization problem:

$$\max_{u_c} \max_{U_2} \left(\sum_{j \in \Gamma_2} K_j(U_2, u_c) + K_c^2(U_2, u_c) \right) =$$

$$= \sum_{j \in \Gamma_2} K_j(\widehat{U}_2, \widehat{u}_c) + K_c^2(\widehat{U}_2, \widehat{u}_c) = 41$$

In this example $\hat{u}_c = B$ next we find $\widetilde{U}_1$ it is the solution of:

$$\max_{U_1} \left(\sum_{j \in \Gamma_i} K_j(U_1, B) + K_c^1(U_1, B) \right) = 33$$

Thus $V(H_1) = 33$

$$V(H_2) = \sum_{j \in \Gamma_2} K_j(\widetilde{U}_2, \hat{u}_c) + K_c^2(\widetilde{U}_2, \hat{u}_c),$$

where $\hat{u}_c$ the solution of this maximization problem:

$$\max_{u_c} \max_{U_1} \left(\sum_{j \in \Gamma_1} K_j(U_1, u_c) + K_c^1(U_1, u_c) \right) =$$

$$= \sum_{j \in \Gamma_1} K_j(\widehat{U}_1, \widehat{u}_c) + K_c^1(\widehat{U}_1, \widehat{u}_c) = 35$$

In this example $\hat{u}_c = A$ next we find $\widetilde{U}_2$ it is the solution of:

$$\max_{U_2} \left(\sum_{j \in \Gamma_2} K_j(U_2, A) + K_c^1(U_2, A) \right) = 31$$

Thus $V(H_2) = 31$

$$V(\mathcal{H}) = \max_{u_c} \max_{U_i} \left(\sum_{i:\ H_i \in \mathcal{H}} \sum_{j \in \Gamma_i} K_j(U_i, u_c) + \sum_{i:\ H_i \in \mathcal{H}} K_c^i(U_i, u_c) \right) = 74$$

Now we use the solution with equal excess to get payoffs for hyperlinks

$$\xi_{H_j} = V(H_i) + \frac{V(\mathcal{H}) - \sum\limits_{i \in \mathcal{H}} V(H_i)}{L}, \quad j = \overline{1, L}.$$

$$\xi_{H_1} = V(H_1) + \frac{V(\mathcal{H}) - (V(H_1) + V(H_2))}{2} = 38$$

$$\xi_{H_2} = V(H_2) + \frac{V(\mathcal{H}) - (V(H_1) + V(H_2))}{2} = 36$$

Second step. Now we solve two cooperative game as an optimality principle we will use proportional solution. For the game on hyperlink H_1 a characteristic function for players 1,2 and c

$$v^1(1) = \max_{u^1} \min_{u_c \bigcup U_1 \setminus u^1} K_i(U_1, u_c) = \max_{u^1} \min_{u_c, u^2} (K_1(u^1, u_c) + K_1(u^1, u^2)) = 10.$$

$$v^1(2) = \max_{u^2} \min_{u_c \bigcup U_1 \setminus u^2} K_i(U_1, u_c) = \max_{u^2} \min_{u_c, u^1} (K_2(u^2, u_c) + K_2(u^1, u^2)) = 6.$$

$$v^1(c) = \max_{u_c} \min_{U_1} K_c^1(U_1, u_c) = \max_{u_c} \min_{u^1, u^2} (K_c^1(u^2, u_c) + K_c^1(u^1, u_c)) = 11$$

$$v^1(N_1) = \xi_{H_1} = 38$$

$$\mathcal{E}_1^1 = \frac{v^1(1)}{v^1(1) + v^1(2) + v^1(c)} v^1(N_1) = \frac{380}{27}$$

$$\mathcal{E}_1^2 = \frac{v^1(2)}{v^1(1) + v^1(2) + v^1(c)} v^1(N_1) = \frac{228}{27}$$

$$\mathcal{E}_1^c = \frac{v^1(c)}{v^1(1) + v^1(2) + v^1(c)} v^1(N_1) = \frac{418}{27}$$

For the game on hyperlink H_2 a characteristic function for players 3,4 and c

$$v^2(3) = \max_{u^3} \min_{u_c \bigcup U_2 \setminus u^3} K_i(U_2, u_c) = \max_{u^3} \min_{u_c, u^4} (K_3(u^3, u_c) + K_3(u^3, u^4)) = 15.$$

$$v^2(4) = \max_{u^4} \min_{u_c \bigcup U_2 \setminus u^4} K_i(U_2, u_c) = \max_{u^2} \min_{u_c, u^3} (K_4(u^4, u_c) + K_4(u^3, u^4)) = 11.$$

$$v^2(c) = \max_{u_c} \min_{U_2} K_c^2(U_2, u_c) = \max_{u_c} \min_{u^1, u^2} (K_c^2(u^3, u_c) + K_c^2(u^4, u_c)) = 8$$

$$v^2(N_2) = \xi_{H_2} = 36$$

$$\mathcal{E}_2^3 = \frac{v^2(3)}{v^2(3) + v^2(4) + v^2(c)} v^2(N_2) = \frac{540}{34}$$

$$\mathcal{E}_2^4 = \frac{v^2(4)}{v^1(3) + v^2(4) + v^2(c)} v^2(N_2) = \frac{396}{34}$$

$$\mathcal{E}_2^c = \frac{v^2(c)}{v^1(3) + v^2(4) + v^2(c)} v^2(N_2) = \frac{288}{34}$$

Third step. Now we sum the payoffs of central player from each hyperlink

$$\mathcal{E}_c = \sum_{j=1}^{L} \mathcal{E}_c^j = \mathcal{E}_1^c + \mathcal{E}_2^c = \frac{288}{34} + \frac{418}{27}$$

3. Generalization of the game

3.1. Preliminaries

In this part we consider the generalization of the previous game. Now we need to refresh some information about hypergraph.

The reduction of hypergraph $(N, \mathscr{H})$ is called hypergraph $(N, \mathscr{H}')$ which is obtained from the original by removing all hyperlinks that are completely contained in other hyperlinks. Hypergraph is called reduced if it is equivalent to its reduction, that is, it does not have a hyperlink inside other hyperlinks.

A simple cycle with length s in hypergraph (N, H) is a sequence

$$(H_0, n_0, H_1, \ldots, H_{s1}, n_{s1}, H_s),$$

where $H_0, \ldots, H_{s1}$ different hyperlinks, hyperlink H_s coincides with $H_0, n_0, \ldots, n_{s-1}$ different vertexes, and $n_i \in H_i \cap H_{i+1}$ for all $i = 0, \ldots, s-1$.

A first definition of acyclicity for hypergraphs was given in Berge, 1989. A hypergraph is acyclic if its incidence graph is acyclic.

3.2. Definition of the game

In this part, we will construct the game where communication structure defined by acyclic reduced hypergraph (N, H). An interpretation of this communication structure can be that there are managers who work with companies and each company has workers who work just on them.

Let $N := \{1, \ldots, n-m, c_1, \ldots, c_m\}$ be a set of players. Denote the numbers of hyperlinks in communication structure by L as before. The players which included just in one hyperlink we will call simple-players, other will be called complex-players. To construct the game we need to introduce new notations.

Let $\mathfrak{u}^i$ is strategy of simple-player i from the set of his strategies $\mathfrak{U}^i$. Also denote as $\mathfrak{u}^{c_j}$ a strategy of complex-player j from the set of his strategies $\mathfrak{U}^{c_j}$. The set of simple-players strategies in hyperlink H_i we will denote as U_i and the set of complex-players strategies in this hyperlink as U_i^c. The payoff function for simple-player j in hyperlink H_i denote as $K^j(U_i, U_i^c)$, and the payoff function for complex-player j in hyperlink H_i denote by $K_i^{c_j}(U_i, U_i^c)$. Now we can define the total payoff function of each player. For all simple-players for example j which included in hyperlink the payoff is equal to

$$h^j = K^j(U_i, U_i^c)$$

the total payoff function of complex-player j we define as

$$h^{c_j} = \sum_{i: c_j \in H_i} K_i^{c_j}(U_i, U_i^c)$$

3.3. Cooperation

Now we consider a cooperative game where the players agree to choose their strategies together to maximize the total sum of theirs payoffs. The total sum is equal:

$$\sum_{i=1}^{n-m} h^i + \sum_{i=1}^{m} h^{c_i} = \sum_{i=1}^{L} \sum_{j \in H_i} K^j(U_i, U_i^c) + \sum_{i=1}^{L} \sum_{j: c_j \in H_i} K_i^{c_j}(U_i, U_i^c)$$

Firstly we consider the cooperative game where players are hyperlinks. We define a characteristic function for any coalition of hyperlinks, after that we use an allocation rule and get payoff for each hyperlink. Next step is consider L cooperative games and get payoff for each player. Finally we find a total payoffs for any complex-player as a sum from his payoffs from each hyperlink in which he exist.

First step We consider the cooperative game with hyperlinks as players. To define the characteristic function for all coalitions we need to introduce new notations. Let Γ_i be a set of simple-players in hyperlink H_i, Γ_i^c is a set of complex-players in hyperlink H_i. Also denote a set of hyperlinks which include complex-player j as B_{c_j}. For any coalition S we will make a partition on each hyperlink in S of complex-player set in this hyperlink. The set of strategies in hyperlink H_i of complex-players which included only in hyperlinks from the coalition S we denote as $U_i^{c^f}$ others by $U_i^{c^n}$, $U_i^{c^f} \cup U_i^{c^n} = U_i^c$. The set of hyperlinks we denote as $\mathcal{H}$. $\mathcal{S} \subseteq \mathcal{H}$ is coalition from this set of hyperlinks. Now we can define the characteristic function for all coalitions of hyperlinks as follows

$$V(\mathcal{S}) = \sum_{i: H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i: H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n})$$

where $\hat{U}_i^{c^n}$ the solution of this maximization problem:

$$\max_{U_i^{c^n}} \max_{U_i} \left(\sum_{i: H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^n}) + \sum_{i: H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^n}) \right) =$$
$$= \sum_{i: H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widehat{U_i}, \widehat{U}_i^{c^n}) + \sum_{i: H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widehat{U_i}, \widehat{U}_i^{c^n})$$

and $\widetilde{U}_i$ and $\widetilde{U}_i^{c^f}$ the solution of:

$$\max_{U_i^{c^f}} \max_{U_i} \left(\sum_{i: H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i: H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) \right) =$$
$$= \sum_{i: H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i: H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n})$$

for the grand coalition we have:

$$V(\mathcal{H}) = \max_{U_i^c} \max_{U_i} \left(\sum_{i: H_i \in \mathcal{H}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^c,) + \sum_{i: H_i \in \mathcal{H}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^c) \right)$$

As allocation rule here we use Shapley value. Notice that in this step we can use any allocation rule from classic cooperative theory.

$$\phi_{H_i}(V) = \sum_{\mathcal{S}\subseteq\mathcal{H}\setminus H_i} \frac{|\mathcal{S}|!(|\mathcal{N}|-|\mathcal{S}|-1)!}{\mathcal{N}!}(V(\mathcal{S}\cup H_i) - V(\mathcal{S}))$$

Second step From the previous step we get payoffs for each hyperlink in our hypergraph. Now we consider a cooperative game on each hyperlink with players which included in it. It means that now we have L independent cooperative games. As an optimality principle we use proportional solution. We denote the set of players on hyperlink H_j by N_j. Assume that $v^j(N_j) = \phi_{H_j}$. Define the value of the characteristic function for each simple players on hyperlink H_j as

$$v^j(i) = \max_{u^i} \min_{U_j^c \bigcup U_j \setminus u^i} K^i(U_j, U_j^c).$$

for complex-player i on the same hyperlink

$$v^j(c_i) = \max_{u^{c_i}} \min_{U_j^c \setminus u^{c_i} \bigcup U_j} K_j^{c_i}(U_j, U_j^c).$$

Now we can define the payoff of each simple-player on the hyperlink H_j as:

$$\mathcal{E}_i^j = \frac{v^j(i)}{\sum\limits_{k\in N_j} v^j(k)} v^j(N_j) = \frac{v^j(i)}{\sum\limits_{k\in N_j} v^j(k)} \phi_{H_i}.$$

The payoff of complex-player i on the hyperlink H_j we will define by

$$\mathcal{E}_{c_i}^j = \frac{v^j(c_i)}{\sum\limits_{k\in N_j} v^j(k)} v^j(N_j) = \frac{v^j(c_i)}{\sum\limits_{k\in N_j} v^j(k)} \phi_{H_i}.$$

Third step Now we get total payoffs for each simple-player. The total payoff for each complex-player is the sum of his payoffs from each hyperlink in which it is included.

$$\mathcal{E}_{c_i} = \sum_{j:H_j\in B_{c_i}} \mathcal{E}_{c_i}^j$$

3.4. Example

Consider the cooperative game with player set $N = \{1, 2, 3, 4, c_1, c_2\}$ and hypergraph $H_1 = \{1, 2, c_1\}, H_2 = \{3, c_1, c_2\}, H_3 = \{c_2, 4\}$ which is shown on fig.3.

For this example we consider that in each hyperlink players have bimatrix game between each other. It means that for simple-player i in hyperlink H_j payoff function is

$$h^i(U_j, U_j^c) = K^i(U_j, U_j^c) = \sum_{k:u^k\in U_j\setminus u^i} K^i(u^k, u^i) + \sum_{k:u^{c_k}\in U_j^c} K^i(u^i, u^{c_k}),$$

and for central player the payoff function

$$h^{c_j} = \sum_{i:c_j\in H_i} K_i^{c_j}(U_i, U_i^c)$$

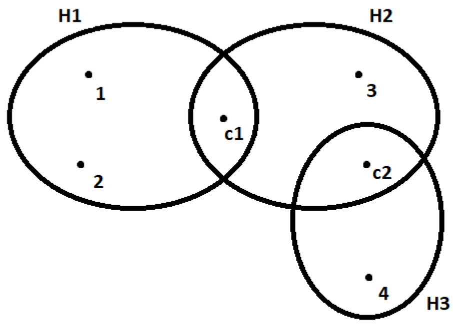

Fig. 3. Communication structure

where

$$K_j^{c_i}(U_j, U_j^c) = \sum_{k:u^k \in U_j} K^{c_i}(u^k, u^{c_i}) + \sum_{k:u^{c_k} \in U_j^c \setminus u^{c_i}} K^{c_i}(u^{c_i}, u^{c_k})$$

Define the bimatrix game for each pair of linked players. We will write a bimatrix 2×2 for player i and j where i chooses the row and j chooses column. We consider that all players have a set of strategies (A, B).

For players 1 and c_1

$$\begin{pmatrix} 4\backslash 8 & 3\backslash 6 \\ 1\backslash 3 & 5\backslash 6 \end{pmatrix}$$

For players 2 and c_1

$$\begin{pmatrix} 3\backslash 6 & 5\backslash 5 \\ 0\backslash 2 & 4\backslash 8 \end{pmatrix}$$

For players 1 and 2

$$\begin{pmatrix} 6\backslash 8 & 6\backslash 0 \\ 4\backslash 3 & 0\backslash 6 \end{pmatrix}$$

For players 3 and c_1

$$\begin{pmatrix} 8\backslash 0 & 6\backslash 10 \\ 3\backslash 6 & 9\backslash 3 \end{pmatrix}$$

For players c_2 and c_1

$$\begin{pmatrix} 5\backslash 2 & 8\backslash 9 \\ 7\backslash 2 & 6\backslash 5 \end{pmatrix}$$

For players 3 and c_2

$$\begin{pmatrix} 0\backslash 1 & 10\backslash 4 \\ 7\backslash 0 & 3\backslash 8 \end{pmatrix}$$

For players 4 and c_2

$$\begin{pmatrix} 1\backslash 4 & 2\backslash 7 \\ 4\backslash 0 & 3\backslash 5 \end{pmatrix}$$

First step. Firstly we will find the value of the characteristic function for all coalitions of hyperlinks. In coalition $\mathcal{S} = \{H_1\}$, $U_1^{c^n} = U_1^c = (u_1^c)$ then the value of characteristic function of this coalition is equal

$$V(H_1) = \sum_{j \in \Gamma_1} K^j(\widetilde{U_1}, \widehat{U}_1^{c^n}) + \sum_{j \in \Gamma_1^c} K_1^{c_j}(\widetilde{U}_1, \widehat{U}_1^{c^n})$$

$$\max_{U_i^{c^n}} \max_{U_i} \left(\sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widehat{U_i}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widehat{U_i}, \widehat{U}_i^{c^n}) = 50$$

from this we get $\widehat{U}_1^{c^n} = (\widehat{u}^{c_1}) = B$

$$\max_{U_1} \left(\sum_{j \in \Gamma_1} K^j(U_1, \widehat{U}_1^{c^n}) + \sum_{j \in \Gamma_1^c} K_1^{c_j}(U_1, \widehat{U}_1^{c^n}) \right) =$$
$$= \sum_{j \in \Gamma_1} K^j(\widetilde{U_1}, \widehat{U}_1^{c^n}) + \sum_{j \in \Gamma_1^c} K_1^{c_j}(\widetilde{U}_1, \widehat{U}_1^{c^n}) = 33$$

Thus we get $V(H_1) = 33$.

In coalition $\mathcal{S} = \{H_2\}$, $U_2^{c^n} = U_2^c = (u^{c_1}, u^{c_2})$ then the value of characteristic function of this coalition is equal

$$V(H_2) = \sum_{j \in \Gamma_2} K^j(\widetilde{U_2}, \widehat{U}_2^{c^n}) + \sum_{j \in \Gamma_2^c} K_2^{c_j}(\widetilde{U}_2, \widehat{U}_2^{c^n})$$

$$\max_{U_i^{c^n}} \max_{U_i} \left(\sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widehat{U_i}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widehat{U_i}, \widehat{U}_i^{c^n}) = 44$$

from this we get $\widehat{U}_2^{c^n} = (\widehat{u}^{c_1}, \widehat{u}^{c_2}) = (A, B)$

$$\max_{U_2} \left(\sum_{j \in \Gamma_2} K^j(U_2, \widehat{U}_2^{c^n}) + \sum_{j \in \Gamma_2^c} K_2^{c_j}(U_2, \widehat{U}_2^{c^n}) \right) =$$
$$= \sum_{j \in \Gamma_2} K^j(\widetilde{U_2}, \widehat{U}_2^{c^n}) + \sum_{j \in \Gamma_2^c} K_i^{c_j}(\widetilde{U}_2, \widehat{U}_2^{c^n}) = 31$$

Thus we get $V(H_2) = 31$.

In coalition $\mathcal{S} = \{H_3\}$, $U_3^{c^n} = U_3^c = (u^{c_2})$ then the value of characteristic function of this coalition is equal

$$V(H_3) = \sum_{j \in \Gamma_3} K^j(\widetilde{U_3}, \widehat{U}_3^{c^n}) + \sum_{j \in \Gamma_3^c} K_3^{c_j}(\widetilde{U}_3, \widehat{U}_3^{c^n})$$

$$\max_{U_i^{c^n}} \max_{U_i} \left(\sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widehat{U_i}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widehat{U_i}, \widehat{U}_i^{c^n}) = 74$$

from this we get $\widehat{U}_3^{c^n} = (\widehat{u}^{c_2}) = (B)$

$$\max_{U_3} \left(\sum_{j \in \Gamma_3} K^j(U_3, \widehat{U}_3^{c^n}) + \sum_{j \in \Gamma_3^c} K_3^{c_j}(U_3, \widehat{U}_3^{c^n}) \right) =$$
$$= \sum_{j \in \Gamma_3} K^j(\widetilde{U_3}, \widehat{U}_3^{c^n}) + \sum_{j \in \Gamma_3^c} K_i^{c_j}(\widetilde{U}_3, \widehat{U}_3^{c^n}) = 9$$

Thus we get $V(H_3) = 9$.

In coalition $\mathcal{S} = \{H_1, H_2\}$, $U_1^{c^f} = U_1^c = (u^{c_1}), U_2^{c^n} = (u^{c_2}), U_2^{c^f} = (u^{c_1})$ then the value of characteristic function of this coalition is equal

$$V(\mathcal{S}) = V(\{H_1, H_2\}) = \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n})$$

$$\max_{U_i^{c^n}} \max_{U_i} \left(\sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widehat{U_i}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widehat{U_i}, \widehat{U}_i^{c^n}) = 9$$

From this we get $\widehat{U}_2^{c^n} = (u^{c_2}) = B$

$$\max_{U_i^{c^f}} \max_{U_i} \left(\sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) = 74$$

Thus we get $V(\{H_1, H_2\}) = 74$.

In coalition $\mathcal{S} = \{H_2, H_3\}$, $U_3^{c^f} = U_3^c = (u^{c_3}), U_2^{c^n} = (u^{c_1}), U_2^{c^f} = (u^{c_2})$ then the value of characteristic function of this coalition is equal

$$V(\mathcal{S}) = V(\{H_2, H_3\}) = \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n})$$

$$\max_{U_i^{c^n}} \max_{U_i} \left(\sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widehat{U_i}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widehat{U_i}, \widehat{U}_i^{c^n}) = 41$$

From this we get $\widehat{U}_2^{c^n} = (u^{c_1}) = A$

$$\max_{U_i^{c^f}} \max_{U_i} \left(\sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) = 40$$

Thus we get $V(\{H_2, H_3\}) = 40$.

In coalition $\mathcal{S} = \{H_1, H_3\}$, $U_1^{c^n} = U_1^c = (u^{c_1}), U_3^{c^n} = U_3^c = (u^{c_2})$ then the value of characteristic function of this coalition is equal

$$V(\mathcal{S}) = V(\{H_1, H_3\}) = \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n})$$

$$\max_{U_i^{c^n}} \max_{U_i} \left(\sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widehat{U_i}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \notin \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widehat{U_i}, \widehat{U}_i^{c^n}) = 35$$

From this we get $\widehat{U}_3^{c^n} = (u^{c_2}) = B$, and $\widehat{U}_1^{c^n} = (u^{c_1}) = B$

$$\max_{U_i^{c^f}} \max_{U_i} \left(\sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^{c^f}, \widehat{U}_i^{c^n}) \right) =$$
$$= \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i} K^j(\widetilde{U_i}, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) + \sum_{i:H_i \in \mathcal{S}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(\widetilde{U}_i, \widetilde{U}_i^{c^f}, \widehat{U}_i^{c^n}) = 42$$

Thus we get $V(\{H_1, H_3\}) = 42$.

For the grand coalition $\mathcal{H}$ the value of characteristic function is equal

$$V(\mathcal{H}) = \max_{U_i^c} \max_{U_i} \left(\sum_{i:H_i \in \mathcal{H}} \sum_{j \in \Gamma_i} K^j(U_i, U_i^c,) + \sum_{i:H_i \in \mathcal{H}} \sum_{j \in \Gamma_i^c} K_i^{c_j}(U_i, U_i^c) \right) = 83$$

In this example at this step we will use the solution with equal excess as an optimality principle.

$$\xi_{H_j} = V(H_i) + \frac{V(\mathcal{H}) - \sum\limits_{i \in \mathcal{H}} V(H_i)}{L}, \quad j = \overline{1, L}.$$

$$\xi_{H_1} = V(H_1) + \frac{V(\mathcal{H}) - (V(H_1) + V(H_2) + V(H_3))}{3} = 36.(3)$$

$$\xi_{H_2} = V(H_2) + \frac{V(\mathcal{H}) - (V(H_1) + V(H_2) + V(H_3))}{3} = 34.(3)$$

$$\xi_{H_3} = V(H_3) + \frac{V(\mathcal{H}) - (V(H_1) + V(H_2) + V(H_3))}{3} = 12.(3)$$

Second step. Now we need to solve three cooperative game as an optimality principle we will use proportional solution. For the game on hyperlink H_1 a characteristic function for players 1,2 and c_1

$$v^j(i) = \max_{u^i} \min_{U_j^c \bigcup U_j \setminus u^i} K^i(U_j, U_j^c).$$

$$v^j(c_i) = \max_{u^{c_i}} \min_{U_j^c \setminus u^{c_i} \bigcup U_j} K_j^{c_i}(U_j, U_j^c).$$

$$v^1(1) = 10, v^1(2) = 6, v^1(c_1) = 11$$

$$v^1(N_1) = \xi_{H_1} = 36.(3)$$

$$\mathcal{E}_1^1 = \frac{v^1(1)}{v^1(1) + v^1(2) + v^1(c_1)} v^1(N_1) = \frac{363.(3)}{27}$$

$$\mathcal{E}_2^1 = \frac{v^1(2)}{v^1(1) + v^1(2) + v^1(c_1)} v^1(N_1) = \frac{218}{27}$$

$$\mathcal{E}_{c_1}^1 = \frac{v^1(c_1)}{v^1(1) + v^1(2) + v^1(c_1)} v^1(N_1) = \frac{399.(6)}{27}$$

For the game on hyperlink H_2 a characteristic function for players 3, c_1 and c_2

$$v^2(3) = 15, v^2(c_1) = 8, v^2(c_2) = 11$$

$$v^2(N_2) = \xi_{H_2} = 34.(3)$$

$$\mathcal{E}_3^2 = \frac{v^2(3)}{v^2(3) + v^2(c_2) + v^2(c_1)} v^2(N_2) = \frac{515}{34}$$

$$\mathcal{E}_{c_1}^2 = \frac{v^2(c_1)}{v^2(3) + v^2(c_2) + v^2(c_1)} v^2(N_2) = \frac{274.(6)}{34}$$

$$\mathcal{E}_{c_2}^2 = \frac{v^2(c_2)}{v^2(3) + v^2(c_2) + v^2(c_1)} v^2(N_2) = \frac{377.(6)}{34}$$

For the game on hyperlink H_3 a characteristic function for players 4 and c_2

$$v^3(4) = 3, v^3(c_2) = 5$$

$$v^3(N_3) = \xi_{H_3} = 12.(3)$$

$$\mathcal{E}_4^3 = \frac{v^3(4)}{v^3(4) + v^3(c_2)} v^3(N_3) = \frac{37}{8}$$

$$\mathcal{E}_{c_2}^3 = \frac{v^3(c_2)}{v^3(4) + v^3(c_2)} v^3(N_3) = \frac{61.(6)}{8}$$

Third step. Now we need to sum the payoffs of players c_1 and c_2

$$\mathcal{E}_{c_1} = \sum_{j:H_j \in B_{c_1}} \mathcal{E}_{c_1}^j = \mathcal{E}_{c_1}^1 + \mathcal{E}_{c_1}^2 = \frac{399.(6)}{27} + \frac{274.(6)}{34}$$

$$\mathcal{E}_{c_2} = \sum_{j:H_j \in B_{c_2}} \mathcal{E}_{c_2}^j = \mathcal{E}_{c_2}^2 + \mathcal{E}_{c_2}^3 = \frac{377.(6)}{34} + \frac{61.(6)}{8}$$

So we get the imputation

$$\mathcal{E}_1 = \frac{363.(3)}{27}, \mathcal{E}_2 = \frac{218}{27}$$

$$\mathcal{E}_3 = \frac{515}{34}, \mathcal{E}_4 = \frac{37}{8}$$

$$\mathcal{E}_{c_1} = \frac{399.(6)}{27} + \frac{274.(6)}{34}, \mathcal{E}_{c_2} = \frac{377.(6)}{34} + \frac{61.(6)}{8}$$

4. Conclusion

A cooperative game with a hypergraph communication structure is proposed. The two-level cooperation in this class of games is considered. A new approach for the definition of the characteristic function for coalitions of hyperlinks is introduced. For a two-level cooperation structure, a new allocation rule is proposed. Examples of hypergraph games are presented.

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