Contributions to Game Theory and Management, XII, 159–176

Solution of the Differential Game with Hybrid Structure*

Ekaterina V. Gromova[1] **and Natalya G. Magnitskaya**[2]

[1] *St. Petersburg State University,*
7/9 Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail: e.v.gromova@spbu.ru
[2] *St. Petersburg State University,*
7/9 Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail: magnitsnatalya@gmail.com

Abstract This paper focuses on two approaches for calculating optimal controls in cooperative differential games with hybrid structure: namely, the (joint) payoff function has a form of sum of integrals with different but adjoint time intervals. Our methods had been applied for the game-theoretical model with random time horizon T where T has a discrete structure. But the area of application can be more wide.
Keywords: differential games, random duration, discontinuous cumulative distribution function, discrete random variable, optimal control, Pontryagin's maximum principle.

1. Introduction

In the paper the particular problem of calculating optimal controls in open-loop form is considered (Pontryagin, 1961). In many continuous optimal control problems including game-theoretic formulation (Basar and Olsder, 1995) in cooperative form the objective functional can be written as an integral from t_0 to T. But in a hybrid formulation (Gromov and Gromova, 2017) where, for example, the payoff can be considered as a sum of integrals with different but adjoint time intervals there is a lack of concrete algorithms for solving the problem. We consider the class of cooperative differential games with discrete random time horizon (Gromova and Tur, 2017; Gromova et al., 2018) to demonstrate the methods which are based on using Pontryagin's maximum principle. The general formulation of the differential games with continuous random time horizon can be found in (Petrosjan and Shevkoplyas, 2000) and the fully discrete case of the dynamic games with discrete random time horizon had been published in (Gromova and Plekhanova, 2019). In the paper we consider hybrid model, namely, continuous dynamics and discontinuous cumulative distribution function which corresponds to discrete random time horizon. Another approach with hybrid cumulative distribution function had been considered in (Gromov and Gromova, 2017).

The paper is structured as follows. In section 2 the problem statement with random discrete time horizon is given, in subsections more particular cases with 1 and 2 points of discontinuity are considered. In section 3 we consider a new approach of solving differential game based on parametrization. In section 5 we analyze another more formal backward approach which includes terminal payoff for results from previous stage, the same example is solved and the results coincide. Numerical example is given in 6. In sections 7, 8 these two methods are applied for the case of

* The reported study was funded by RFBR according to the research project N 18-00-00727 (18-00-00725)

two points of discontinuity and it is shown that results coincides.

2. Game formulation

Consider a differential game with n players (Basar and Olsder, 1995). The game starts at the time t_0 and ends at the random moment T, where T is the random variable with known cumulative distribution function $F(t)$, $t \in [t_0, T_f]$, T_f can be infinite (Gromova and Tur, 2017) or finite (Petrosjan and Murzov, 1966). The random variable T is formed as follows (see similar approach in (Gromova et al., 2016; Kostyunin et al., 2014)). Let T_i be the random variable with known cumulative distribution function $F_i(t)$, $t \in [t_0, T_f]$, $i = \overline{1,n}$. T_i — the time instant of the process stop for the player i, $i = \overline{1,n}$. $\{T_i\}_{i=1}^{n}$ are assumed to be independent random variables. We assume, that game starts at the time t_0 and ends at the time of the game stop for the first player, which means

$$T = min\{T_1, T_2, ...T_n\}. \tag{1}$$

Since $T_1, \ldots, T_n$ are independent, we have:

$$F(t) = 1 - \prod_{l=1}^{n}(1 - F_l(t)). \tag{2}$$

The dynamics is defined by the equation:

$$\dot{x} = g(x, u_1, u_2, ...u_n), \quad x(t_0) = x_0. \tag{3}$$

Payoff of the player i is defined as follows:

$$K_i(t_0, x, u) = E(\int_{t_0}^{T_f} h_i(x(t), u(t))dt), \quad i = \overline{1,n}, \tag{4}$$

where $h_i(x, u_1...u_n)$ is the instantaneous payoff function of the player i.

It was shown in (Kostyunin and Shevkoplyas, 2011) that uder some mild coniditions the payoff (3) can be transformed to the more simple form:

$$K_i(\cdot) = \int_{t_0}^{T_f} (1 - F(t))h_i(\cdot)dt. \tag{5}$$

Below we will consider the case of discrete random variables $T_i, i = 1, \ldots, n$. It means that for all players there are known time instants $\{t_1, t_2, ..., t_k\}$ in which the game may stop with some probabilities $\{p_1, p_2, ..., p_k\}$.

For the simplicity let us consider the case of 2 players. Let T_1 and T_2 be discrete random variables with known cumulative distribution functions $F_1(t)$ and $F_2(t)$. Let $\{t_1, t_2, ..., t_k\}$ and $\{\tau_1, \tau_2, ..., \tau_k\}$ be time instants, when distribution functions $F_1(t)$, $F_2(t)$ have simple discontinuity. Let

$$P\{T_1 = t_m\} = p_m, \quad P\{T_2 = \tau_j\} = \varrho_j, \quad m, j = \overline{1,k}.$$

It is assumed, that the game ends at the time of the game stop for the first player: $T = \min\{T_1, T_2\}$. Since T_1, T_2 are independent, hense:

$$F(t) = 1 - \prod_{l=1}^{2}(1 - F_l(t)). \tag{6}$$

2.1. Problem statement for one point of discontinuity

Consider the following case: $F_1(t)$, $F_2(t)$ have simple discontinuities in points $t_1 = \tau_1$, $t_2 = \tau_2$ (Fig. 1., Fig. 2.).

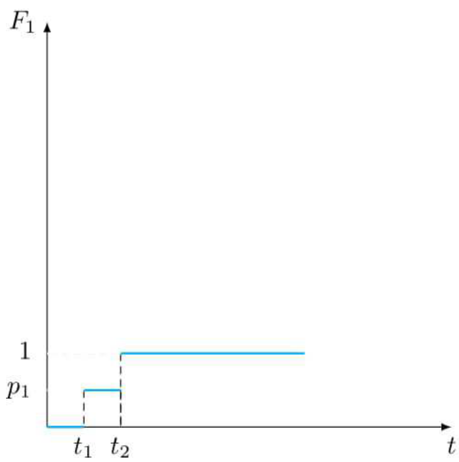

Fig. 1. c.d.f. F_1

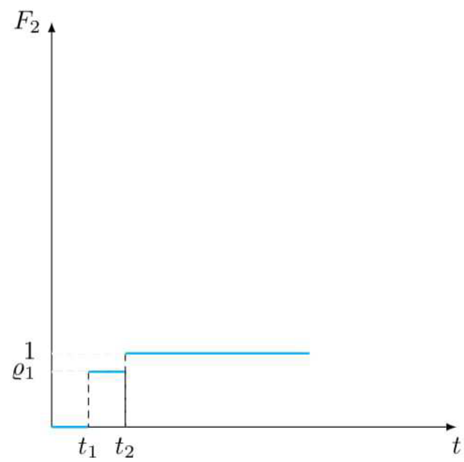

Fig. 2. c.d.f. F_2

According to (6), cumulative distribution function of the random variable T is:

$$F(t) = \begin{cases} 0 & t < t_1, \\ \rho_1 = 1 - (1 - p_1)(1 - \varrho_1) & t_1 \le t < t_2, \\ 1 & t \ge t_2. \end{cases}$$

From (5) for discrete random time horizon (see (Gromova and Tur, 2017)), we have payoff of the player i, $i = 1, 2$ in the following form:

$$K_i(\cdot) = \int_{t_0}^{t_1} h_i(x(t), u(t))dt + (1 - \rho_1) \int_{t_1}^{t_2} h_i(x(t), u(t))dt.$$

Let $u(t) = (u_1(t), u_2(t))$.

Consider the cooperative form of the game (Petrosjan and Danilov, 1982). Then the optimal control problem is to maximize the total payoff of the players:

$$\begin{aligned} &\max_{u_1,u_2} \textstyle\sum_{i=1}^{2} K_i(t_0, x_0, u_1, u_2) = \int_{t_0}^{t_1}(h_1(x^*(t), u^*(t)) + h_2(x^*(t), u^*(t))dt + \\ &+(1 - \rho_1)\int_{t_1}^{t_2}(h_1(x^*(t), u^*(t)) + h_2(x^*(t), u^*(t))dt, \end{aligned} \tag{7}$$

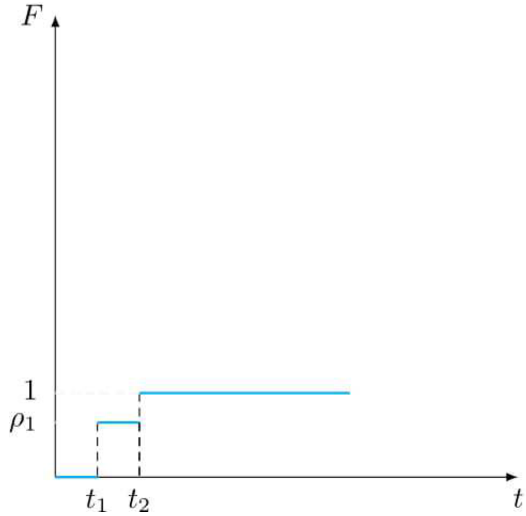

Fig. 3. c.d.f. $F(t)$

where $x^*(t),\ u^*(t)$ — optimal trajectory and controls.
The solution will be considered in the class of open-loop strategies (Afanasyev et al., 2003).

2.2. Problem statement for two points of discontinuity

Consider more complicated case of discrete distribution. Let $F_1(t)$, $F_2(t)$ have simple discontinuities in points $t_1 = \tau_1$, $t_2 = \tau_2$, $t_3 = \tau_3$ (Fig. 4., Fig 5.).

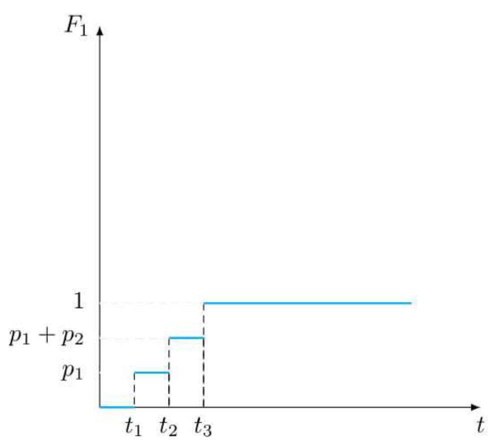

Fig. 4. c.d.f. F_1

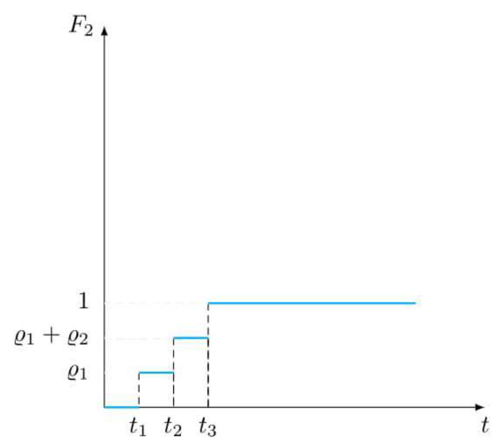

Fig. 5. c.d.f. F_2

Then

$$F(t) = 1 - \prod_{i=1}^{2}(1 - F_i) = 1 - (1 - F_1)(1 - F_2),$$

$$F(t) = \begin{cases} 0 & t \le t_1, \\ \rho_1 = 1 - (1-p_1)(1-\varrho_1) & t_1 < t \le t_2, \\ \rho_2 = 1 - (1-p_1-p_2)(1-\varrho_1-\varrho_2) & t_2 < t \le t_3, \\ 1 & t > t_3. \end{cases}$$

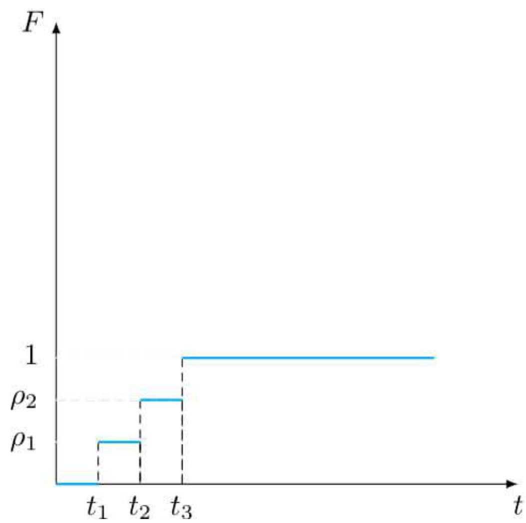

Fig. 6. c.d.f. $F(t)$

From (5) and (Gromova and Tur, 2017), we get payoff of the player $i,\ i = 1, 2$:

$$K_i(\cdot) = \int_{t_0}^{t_1} h_i(x(t), u(t))dt + (1-\rho_1)\int_{t_1}^{t_2} h_i(x(t), u(t))dt + (1-\rho_2)\int_{t_2}^{t_3} h_i(x(t), u(t))dt. \tag{8}$$

Consider the cooperative form of the game. Then the optimal control problem is to maximize the total payoff of the players:

$$\begin{aligned} &\max_{u_1,u_2} \textstyle\sum_{i=1}^{2} K_i(t_0, x_0, u_1, u_2) = \int_{t_0}^{t_1}(h_1(x^*(t), u^*(t)) + h_2(x^*(t), u^*(t))dt + \\ &+(1-\rho_1)\int_{t_1}^{t_2}(h_1(x^*(t), u^*(t)) + h_2(x^*(t), u^*(t))dt + \\ &+(1-\rho_2)\int_{t_2}^{t_3}(h_1(x^*(t), u^*(t)) + h_2(x^*(t), u^*(t))dt, \end{aligned} \tag{9}$$

where $x^*(t),\ u^*(t)$ — optimal trajectory and controls.

3. First approach. One point of discountinuity

Let us demonstrate the first approach of calculation open-loop controls for (7) by the example of resource extraction differential game (Gromova, 2016) based on models (Breton et al., 2005; Haurie et al., 2012; Jørgensen and Zaccour, 2007):

The dynamics of total amount of resource x is defined by the equation:

$$\dot{x}(t) = -\sum_{i=1}^{2} u_i(t), \quad x(t_0) = x_0. \tag{10}$$

The instantaneous payoff of i-th player is defined as:

$$h_i(x,u) = r_i(u) - d_i x, \quad r_i(u_i) = u_i(a_i - \frac{1}{2}u_i), a_i, d_i > 0, \quad \forall i = 1,2. \tag{11}$$

The solution will be considered in the class of open-loop strategies (Afanasyev et al., 2003). Let $u(t) = (u_1(t), u_2(t))$.
We will apply the Pontryagin's maximum principle (Pontryagin, 1961) and find the solution on two intervals $I_1 = [0, t_1]$ and $I_2 = [t_1, t_2]$, the problem will be solved with two fixed ends on the first interval, and at the second — with a loose right end. We introduce $x(t_1) = x_1$ as a parameter of the solution, we will find it's value at the end of the solution from the maximization condition (7).
Interval I_1.
To find the profile of optimal controls and trajectory we have to solve the maximization problem $\int\limits_{t_0}^{t_1}(h_1(x(t), u(t)) + h_2(x(t), u(t))dt$ for dynamic (10) and initial conditions $x(t_0) = x_0, x(t_1) = x_1$, where x_1 — parameter.
The Hamiltonian is:

$$H(x,u,\psi) = -\psi \sum_{i=1}^{2} u_i + h_1(x(t), u(t)) + h_2(x(t), u(t)) =$$

$$-\psi \sum_{i=1}^{2} u_i + u_1(a_1 - \frac{1}{2}u_1) + u_2(a_2 - \frac{1}{2}u_2) - d_1 x - d_2 x, \tag{12}$$

its first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H}{\partial u_i} = -\psi + (a_i - u_i) = 0,$$

$$u_i^*(t) = -\psi + a_i.$$

The Hessian matrix is negative definite hence we conclude that Hamiltonian H is concave w.r.t. u_i., $t \in [0, t_1]$,

$$\frac{\partial^2 H}{\partial u_i^2} = -1 < 0.$$

The adjoint equations:

$$\frac{\partial \psi}{\partial t} = -\frac{\partial H(x,u,\psi)}{\partial x} = \hat{d}, \quad \hat{d} = d_1 + d_2. \tag{13}$$

Hence,

$$\psi(t) = \psi_0 + \hat{d}t,$$

$$u_i^*(t) = -\psi_0 - \hat{d}t + a_i.$$

Dynamic is:

$$\dot{x}(t) = -\sum_{i=1}^{2} u_i(t) = 2\psi_0 + 2\hat{d}t - \hat{a}, \quad \hat{a} = a_1 + a_2.$$

We use the initial conditions:

$$x(0) = x_0, \quad x(t_1) = x_1, \tag{14}$$

$$x(t) = 2\psi_0 t + \hat{d}t^2 - \hat{a}t + x_0.$$

Let us find ψ_0 according to the initial condition and $x(t)$:

$$\psi_0 = \frac{x_1 - \hat{d}t_1^2 + \hat{a}t_1 - x_0}{2t_1}.$$

Then the optimal trajectory for the interval I_1:

$$x^*(t)_{I_1} = \frac{x_1 - \hat{d}t_1^2 + \hat{a}t_1 - x_0}{t_1} t + \hat{d}t^2 - \hat{a}t + x_0 = (x_1 - x_0)\frac{t}{t_1} + \hat{d}t(t - t_1) + x_0. \tag{15}$$

Optimal player controls for the interval I_1:

$$u_i^*(t)_{I_1} = -\frac{x_1 - \hat{d}t_1^2 + \hat{a}t_1 - x_0}{2t_1} - \hat{d}t + a_i. \tag{16}$$

Interval I_2.
To find the profile of optimal controls and trajectory we have to solve the maximization problem $\int_{t_1}^{t_2}(h_1(x(t), u(t)) + h_2(x(t), u(t))dt$ for dynamic (10) and initial condition $x(t_1) = x_1$, where x_1 — parameter. The Hamiltonian is:

$$H(x, u, \psi) = -\psi \sum_{i=1}^{2} u_i + (1 - \rho_1)h_1(x(t), u(t)) + (1 - \rho_1)h_2(x(t), u(t)) =$$

$$-\psi \textstyle\sum_{i=1}^{2} u_i + (1 - \rho_1)u_1(a_1 - \frac{1}{2}u_1) + (1 - \rho_1)u_2(a_2 - \frac{1}{2}u_2) -$$

$$-(1 - \rho_1)d_1 x - (1 - \rho_1)d_2 x, \tag{17}$$

its first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H}{\partial u_i} = -\psi + (1 - \rho_1)(a_i - u_i) = 0,$$

$$u_i^*(t) = \frac{-\psi + (1 - \rho_1)a_i}{(1 - \rho_1)}.$$

The Hessian matrix is negative definite hence we conclude that Hamiltonian H is concave w.r.t. u_i., $t \in [t_1, t_2]$:

$$\frac{\partial^2 H}{\partial u_i^2} = -(1 - \rho_1) < 0.$$

The adjoint equations:

$$\frac{\partial \psi}{\partial t} = -\frac{\partial H(x,u,\psi)}{\partial x} = (1-\rho_1)\hat{d}, \quad \hat{d} = d_1 + d_2, \tag{18}$$

$$\psi(t) = \int\limits_{t_1}^{t} (1-\rho_1)\hat{d}dt = (1-\rho_1)\hat{d}(t-t_1) + \psi_1.$$

The transversality condition:

$$\psi(t_2) = 0.$$

We get:

$$\psi(t) = (1-\rho_1)\hat{d}(t-t_2).$$

Dynamics:

$$\dot{x}(t) = -\sum_{i=1}^{2} u_i(t) = 2\frac{\psi}{(1-\rho_1)} - \hat{a}, \quad \hat{a} = a_1 + a_2.$$

According to the initial condition: $x(t_1) = x_1$, we get

$$x(t) = -2\hat{d}t_2 t + \hat{d}t^2 - \hat{a}t + 2\hat{d}t_2 t_1 - \hat{d}t_1^2 + \hat{a}t_1 + x_1.$$

Optimal control:

$$u_i^*(t)_{I_2} = -\hat{d}(t-t_2) + a_i. \tag{19}$$

Optimal trajectory:

$$x^*(t)_{I_2} = -2\hat{d}t_2 t + \hat{d}t^2 - \hat{a}t + 2\hat{d}t_2 t_1 - \hat{d}t_1^2 + \hat{a}t_1 + x_1. \tag{20}$$

Intervals I_1, I_2

According to (7) we have to solve the maximization problem, taking into account (15), (16), (20), (19) i.e.

$$\max_{x_1} \sum_{i=1}^{2} K_i(t_0, x_0, u^*(t, x_1)).$$

Substituting (15), (16), (20), (19) into (7), we get:

$$\int\limits_{0}^{t_1} (u_1^*(t)_{I_1}(a_1 - \frac{1}{2}u_1^*(t)_{I_1}) + u_2^*(t)_{I_1}(a_2 - \frac{1}{2}u_2^*(t)_{I_1}) - d_1 x^*(t)_{I_1} - d_2 x^*(t)_{I_1})dt +$$

$$+(1-\rho_1)\int\limits_{t_1}^{t_2}(u_1^*(t)_{I_2}(a_1 - \tfrac{1}{2}u_1^*(t)_{I_2}) + u_2^*(t)_{I_2}(a_2 - \tfrac{1}{2}u_2^*(t)_{I_2}) - d_1 x^*(t)_{I_2} - -d_2 x^*(t)_{I_2})dt =$$

$$= -\frac{x_1^2}{4t_1} + \frac{x_1(-\hat{a}t_1 + x_0)}{2t_1} - \frac{\hat{d}x_1 t_1}{2} + (1-\rho_1)\hat{d}x_1(t_1 - t_2) + C(t_1, t_2), \tag{21}$$

where $C(t_1,t_2)$ — expression independent of x_1.
The maximum (21) is reached at:

$$x_1 = -\hat{a}t_1 + x_0 - \hat{d}t_1^2 + 2t_1(1-\rho_1)\hat{d}(t_1 - t_2).$$

Substituting the obtained value for x_1 into (15), (16), (20), (19), we finally get expressions for the optimal trajectory and controls on the intervals I_1, I_2:

$$x^*_{I_1}(t) = -\hat{a}t + 2(1-\rho_1)\hat{d}(t_1 - t_2)t + \hat{d}t(t - 2t_1) + x_0,$$

$$u^*_{i_{I_1}}(t) = \hat{d}t_1 - (1-\rho_1)\hat{d}(t_1 - t_2) - \hat{d}t + a_i, \quad t \in [t_0; t_1],$$

$$x^*_{I_2}(t) = -2\hat{d}t_2 t + \hat{d}t^2 - \hat{a}t + 2\hat{d}t_2 t_1 - 2\hat{d}t_1^2 + x_0 + \, + 2t_1(1-\rho_1)\hat{d}(t_1 - t_2),$$

$$u^*_{i_{I_2}}(t) = -\hat{d}(t - t_2) + a_i, \quad t \in (t_1; t_2].$$

4. Second approach. One point of discountinuity

Consider the previous example, starting the solution from the second interval I_2. The value $(1-\rho_1)\int\limits_{t_1}^{t_2}(h_1(x(t),u(t)) + h_2(x(t),u(t))dt$ will be considered as the terminal payoff for the interval I_1. Let us substitute (19), (20) and find the value of the terminal payoff:

$\Phi(x_1) = (1-\rho_1)\int\limits_{t_1}^{t_2}(u_1^*(t)_{I_2}(a_1 - \frac{1}{2}u_1^*(t)_{I_2}) + u_2^*(t)_{I_2}(a_2 - \frac{1}{2}u_2^*(t)_{I_2}) - d_1 x^*(t)_{I_2} - d_2 x^*(t)_{I_2})dt =$

$$= \frac{(t_1 - t_2)(\rho_1 - 1)}{6}(3a_1^2 + 3a_2^2 + 2\hat{d}^2(t_1 - t_2)^2 - 3\hat{a}\hat{d}t_1 + 3\hat{a}\hat{d}t_2 - 6x_1\hat{d}). \qquad (22)$$

Interval I_1.

To find the profile of optimal strategies we have to solve the maximization problem $\int\limits_{t_0}^{t_1}(h_1(x(t),u(t)) + h_2(x(t),u(t))dt + \Phi(x_1)$ for dynamic (10), initial condition $x(t_0) = x_0$ and terminal payoff (22). The Hamiltonian is:

$$H(x,u,\psi) = -\psi\sum_{i=1}^{2}u_i + h_1(x(t),u(t)) + h_2(x(t),u(t)) =$$

$$-\psi\sum_{i=1}^{2}u_i + u_1(a_1 - \frac{1}{2}u_1) + u_2(a_2 - \frac{1}{2}u_2) - d_1 x - d_2 x, \qquad (23)$$

its first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H}{\partial u_i} = -\psi + (a_i - u_i) = 0,$$

$$u_i^*(t) = -\psi + a_i.$$

The Hessian matrix is negative definite hence we conclude that Hamiltonian H is concave w.r.t. u_i., $t \in [0, t_1]$:

$$\frac{\partial^2 H}{\partial u_i^2} = -1 < 0.$$

The adjoint equations:

$$\frac{\partial\psi}{\partial t} = -\frac{\partial H(x,u,\psi)}{\partial x} = \hat{d}, \quad \hat{d} = d_1 + d_2. \qquad (24)$$

Hence,

$$\psi(t) = \psi_0 + \hat{d}t,$$

$$u_i^*(t) = -\psi_0 - \hat{d}t + a_i.$$

Dynamics:

$$\dot{x}(t) = -\sum_{i=1}^{2} u_i(t) = 2\psi_0 + 2\hat{d}t - \hat{a}, \quad \hat{a} = a_1 + a_2.$$

We use the initial condition:

$$x(0) = x_0, \tag{25}$$

$$x(t) = 2\psi_0 t + \hat{d}t^2 - \hat{a}t + x_0.$$

According to the terminal payoff condition (22)

$$\psi(t_1) = \frac{\partial \Phi(x_1)}{\partial x_1},$$

$$\psi(t_1) = (t_1 - t_2)(1 - \rho_1)\hat{d}.$$

Then

$$\psi_0 = \hat{d}(-t_1\rho_1 - t_2 + t_2\rho_1).$$

Optimal controls for the I_1 interval:

$$u_i^*(t)_{I_1} = -\hat{d}(-t_1\rho_1 - t_2 + t_2\rho_1) - \hat{d}t + a_i. \tag{26}$$

Optimal trajectory for the interval I_1:

$$x_{I_1}^*(t) = 2\hat{d}(-t_1\rho_1 - t_2 + t_2\rho_1)t + \hat{d}t^2 - \hat{a}t + x_0. \tag{27}$$

Let us substitute the optimal control and trajectory into $\int\limits_{t_0}^{t_1}(h_1(x(t), u(t)) + h_2(x(t), u(t))dt + \Phi(x_1)$ and get:

$$\int\limits_0^{t_1}(u_1^*(t)_{I_1}(a_1 - \frac{1}{2}u_1^*(t)_{I_1}) + u_2^*(t)_{I_1}(a_2 - \frac{1}{2}u_2^*(t)_{I_1}) - d_1 x^*(t)_{I_1} - d_2 x^*(t)_{I_1})dt + \Phi(x_1)$$

$$= \Phi(x_1) + C(t_1, t_2), \tag{28}$$

where $C(t_1, t_2)$— expression independent of x_1.
We know that

$$x_{I_1}^*(t_1) = x_1.$$

Then x_1:

$$x_1 = 2\hat{d}(-t_1\rho_1 - t_2 + t_2\rho_1)t_1 + \hat{d}t_1^2 - \hat{a}t_1 + x_0 = -\hat{a}t_1 + x_0 - \hat{d}t_1^2 + 2t_1(1-\rho_1)\hat{d}(t_1 - t_2). \tag{29}$$

Substitute (29) into (19), (20) and get expressions for optimal trajectories and controls:

$$x^*_{I_1}(t) = 2\hat{d}(-t_1\rho_1 - t_2 + t_2\rho_1)t + \hat{d}t^2 - \hat{a}t + x_0 = \\ -\hat{a}t + 2(1-\rho_1)\hat{d}(t_1 - t_2)t + \ +\hat{d}t(t - 2t_1) + x_0,$$

$$u^*_{i_{I_1}}(t) = -\hat{d}(-t_1\rho_1 - t_2 + t_2\rho_1) - \hat{d}t + a_i = \hat{d}t_1 - (1-\rho_1)\hat{d}(t_1 - t_2) - \hat{d}t + a_i, \quad t \in [t_0; t_1],$$

$$x^*_{I_2}(t) = -2\hat{d}t_2 t + \hat{d}t^2 - \hat{a}t + 2\hat{d}t_2 t_1 - 2\hat{d}t_1^2 + x_0 + 2t_1(1-\rho_1)\hat{d}(t_1 - t_2),$$

$$u^*_{i_{I_2}}(t) = -\hat{d}(t - t_2) + a_i, \quad t \in (t_1; t_2].$$

5. Numerical example

Consider the previous example with numeric parameters.
Let $a_1 = 5$, $a_2 = 6$, $d_1 = 1$, $d_2 = 2$, $p_1 = 0.3$, $\varrho_1 = 0.7$, $\rho_1 = 0.79$, $t_1 = 1$, $t_2 = 2$, $x_0 = 40$.
Consequently:

$$x^*_{I_1}(t) = 3t^2 - 18.26t + 40,$$

$$u^*_{1_{I_1}}(t) = -3t + 8.63,$$

$$u^*_{2_{I_1}}(t) = -3t + 9.63, \quad t \in [t_0; t_1],$$

$$x^*_{I_2}(t) = 3t^2 - 23t + 44.74,$$

$$u^*_{1_{I_2}}(t) = -3t + 11,$$

$$u^*_{2_{I_2}}(t) = -3t + 12, \quad t \in [t_1; t_2].$$

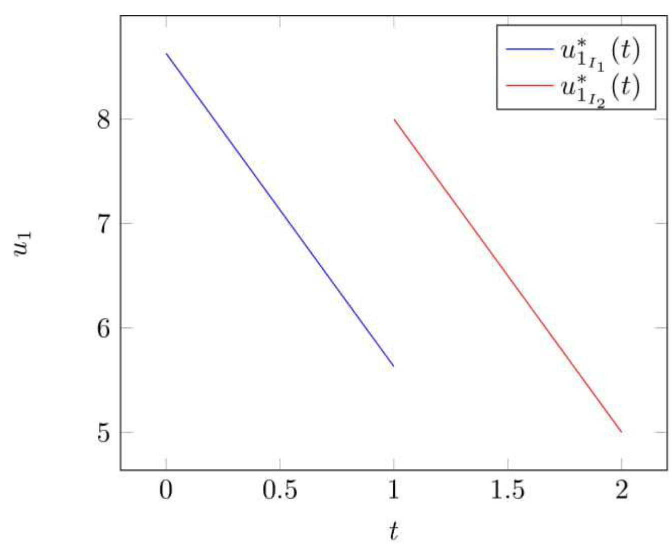

Fig. 7. Optimal control for the first player

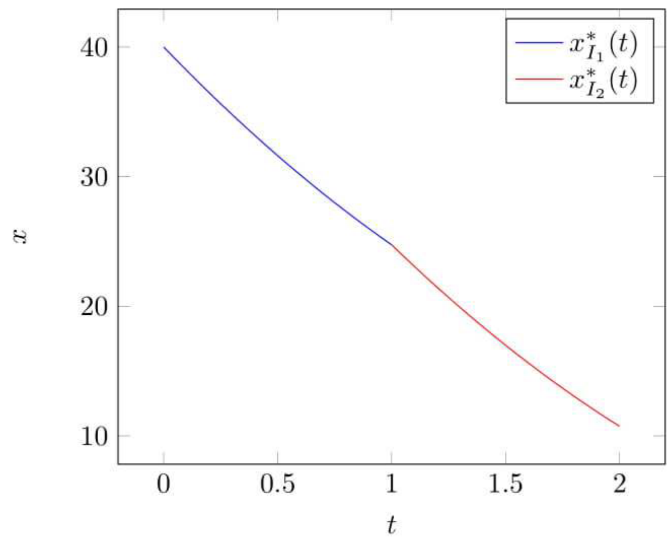

Fig. 8. Optimal trajectory

6. First approach. Two points of discountinuity

Let us demonstrate the first approach of calculation open-loop controls for (9), dynamics (10), instantaneous payoff (11) by the example of resource extraction differential game (Jørgensen and Zaccour, 2007):

We will apply the Pontryagin's maximum principle (Pontryagin, 1961) and find the solution on three intervals $I_1 = [0, t_1]$, $I_2 = [t_1, t_2]$, $I_3 = [t_2, t_3]$, the problem will be solved with two fixed ends on the first and second intervals, and at the third — with a loose right end. We introduce $x(t_1) = x_1$, $x(t_2) = x_2$ as a parameters of the solution, we will find its value at the end of the solution from the maximization condition (9).

Interval I_1.

To find the profile of optimal strategies we have to solve the maximization problem $\int\limits_{t_0}^{t_1}(h_1(x(t), u(t)) + h_2(x(t), u(t))dt$ for dynamic (10) and initial conditions $x(t_0) = x_0$, $x(t_1) = x_1$, where x_1 — parameter.

By using the Pontryagin's maximum principle we get:

optimal trajectory:

$$x^*(t)_{I_1} = \frac{x_1 - \hat{d}t_1^2 + \hat{a}t_1 - x_0}{t_1}t + \hat{d}t^2 - \hat{a}t + x_0 =$$

$$(x_1 - x_0)\frac{t}{t_1} + \hat{d}t(t - t_1) + x_0, \tag{30}$$

optimal controls:

$$u_i^*(t)_{I_1} = -\frac{x_1 - \hat{d}t_1^2 + \hat{a}t_1 - x_0}{2t_1} - \hat{d}t + a_i. \tag{31}$$

Interval I_2.

To find the profile of optimal strategies we have to solve the maximization problem $\int_{t_1}^{t_2}(h_1(x(t),u(t)) + h_2(x(t),u(t))dt$ for dynamic (10) and initial conditions $x(t_1) = x_1$, $x(t_2) = x_2$, where x_1, x_2 — parameters. By using the Pontryagin's maximum principle we get:
optimal trajectory:

$$x^*(t)_{I_2} = \frac{(x_2 - x_1 + \hat{a}(t_2 - t_1) - \hat{d}(t_2^2 - t_1^2))(t - t_1)}{t_2 - t_1} + \hat{d}(t^2 - t_1^2) - \; -\hat{a}(t - t_1) + x_1, \tag{32}$$

optimal controls:

$$u_i^*(t)_{I_2} = -\frac{x_2 - x_1 + \hat{a}(t_2 - t_1) - \hat{d}(t_2^2 - t_1^2)}{2(t_2 - t_1)} - \hat{d}t + a_i. \tag{33}$$

Interval I_3.
To find the profile of optimal strategies we have to solve the maximization problem $\int_{t_2}^{t_3}(h_1(x(t),u(t))+h_2(x(t),u(t))dt$ for dynamic (10) and initial condition $x(t_2) = x_2$, where x_2 — parameter. By using the Pontryagin's maximum principle we get:
optimal controls:

$$u_i^*(t)_{I_3} = -\hat{d}(t - t_3) + a_i, \tag{34}$$

optimal trajectory:

$$x^*(t)_{I_3} = -2\hat{d}t_3t + \hat{d}t^2 - \hat{a}t + 2\hat{d}t_3t_2 - \hat{d}t_2^2 + \hat{a}t_2 + x_2. \tag{35}$$

Intervals I_1, I_2, I_3
According to (9) we have to solve the maximization problem, taking into account (30), (31), (32), (33), (34), (35) i.e.

$$\max_{x_1,x_2} \sum_{i=1}^{2} K_i(t_0, x_0, u^*(t, x_1)).$$

Substituting (30), (31), (32), (33), (34), (35) into (9), we get:

$$\int_0^{t_1} (u_1^*(t)_{I_1}(a_1 - \frac{1}{2}u_1^*(t)_{I_1}) + u_2^*(t)_{I_1}(a_2 - \frac{1}{2}u_2^*(t)_{I_1}) - d_1x^*(t)_{I_1} - d_2x^*(t)_{I_1})dt+$$

$$(1-\rho_1)\int_{t_1}^{t_2} (u_1^*(t)_{I_2}(a_1 - \frac{1}{2}u_1^*(t)_{I_2}) + u_2^*(t)_{I_2}(a_2 - \frac{1}{2}u_2^*(t)_{I_2}) - \hat{d}x^*(t)_{I_2}dt+$$

$$(1-\rho_2)\int_{t_2}^{t_3} (u_1^*(t)_{I_3}(a_1 - \frac{1}{2}u_1^*(t)_{I_3}) + u_2^*(t)_{I_3}(a_2 - \frac{1}{2}u_2^*(t)_{I_3}) - \hat{d}x^*(t)_{I_3}dt =$$

$$= -\frac{x_1^2}{4t_1} + \frac{x_1(-\hat{a}t_1 + x_0)}{2t_1} - \frac{\hat{d}x_1t_1}{2} + (1-\rho_1)\left(-\frac{(-x_2+x_1)^2}{4(t_2-t_1)} + \frac{\hat{a}(-x_2+x_1)}{2}\right) +$$

$$+(1-\rho_1)\left(\frac{\hat{d}(x_2-x_1)(t_1-t_2)}{2}-\hat{d}x_1(t_2-t_1)\right)+(1-\rho_2)\hat{d}x_2(t_2-t_3)+$$
$$+C(t_1,t_2,t_3,x_0), \qquad (36)$$

where $C(t_1,t_2)$ — expression independent of x_1, x_2.
The maximum (36) is reached at:

$$x_1=-\hat{a}t_1+x_0-\hat{d}t_1^2+2t_1(1-\rho_2)\hat{d}(t_2-t_3)+2t_1(1-\rho_1)\hat{d}(t_1-t_2),$$

$$x_2=-\hat{a}(t_2-t_1)-\hat{d}(t_1-t_2)^2+\frac{2(1-\rho_2)\hat{d}(t_2-t_3)(t_2-t_1)}{1-\rho_1}+x_1.$$

Hence,

$$x^*(t)_{I_1}=2t\hat{d}(-t_1+(1-\rho_1)(t_1-t_2)+(1-\rho_2)(t_2-t_3))+\hat{d}t^2-\hat{a}t+x_0,$$

$$u_i^*(t)_{I_1}=\hat{d}(t_1-(1-\rho_1)(t_1-t_2)-(1-\rho_2)(t_2-t_3))-\hat{d}t+a_i, \quad t\in[t_0;t_1],$$

$$x^*(t)_{I_2}=\left(-2\hat{d}t_2+\frac{2(1-\rho_2)\hat{d}(t_2-t_3)}{1-\rho_1}\right)(t-t_1)+\hat{d}t^2-\hat{a}t-2\hat{d}t_1^2+x_0+$$
$$+2t_1(1-\rho_2)\hat{d}(t_2-t_3)+2t_1(1-\rho_1)\hat{d}(t_1-t_2),$$

$$u_i^*(t)_{I_2}=\hat{d}t_2-\frac{(1-\rho_2)\hat{d}(t_2-t_3)}{1-\rho_1}-\hat{d}t+a_i, \quad t\in(t_1;t_2],$$

$$x^*(t)_{I_3}=\hat{d}t^2-\hat{a}t-2\hat{d}t_3t+2\hat{d}t_3t_2-2\hat{d}t_2^2-2\hat{d}t_1^2+2\hat{d}t_1t_2+$$
$$+\frac{2(1-\rho_2)\hat{d}(t_2-t_3)(t_2-t_1)}{1-\rho_1}+x_0+2t_1(1-\rho_2)\hat{d}(t_2-t_3)+2t_1(1-\rho_1)\hat{d}(t_1-t_2),$$

$$u_i^*(t)_{I_3}=-\hat{d}(t-t_3)+a_i, \quad t\in(t_2;t_3].$$

7. Second approach. Two points of discountinuity

Consider the previous example, starting the solution from the third interval I_3. The value $(1-\rho_2)\int\limits_{t_2}^{t_3}(h_1(x(t),u(t))+h_2(x(t),u(t))dt$ will be considered as the terminal payoff for the interval I_2. Let us substitute (34), (35) and find the value of the terminal payoff:
$\Phi_3(x_2)=(1-\rho_2)\int\limits_{t_2}^{t_3}(u_1^*(t)_{I_3}(a_1-\frac{1}{2}u_1^*(t)_{I_3})+u_2^*(t)_{I_3}(a_2-\frac{1}{2}u_2^*(t)_{I_3})-d_1x^*(t)_{I_2}-d_2x^*(t)_{I_2})dt=$

$$=\frac{(t_2-t_3)(\rho_2-1)}{6}(3a_1^2+3a_2^2+2\hat{d}^2(t_2-t_3)^2-3\hat{a}\hat{d}t_2+3\hat{a}\hat{d}t_3-6x_2\hat{d}. \qquad (37)$$

Interval I_2.
To find the profile of optimal strategies we have to solve the maximization problem

$\int_{t_1}^{t_2}(h_1(x(t),u(t))+h_2(x(t),u(t))dt$ for dynamic (10), initial condition $x(t_1)=x_1$ and terminal payoff (37). By using the Pontryagin's maximum principle we get: optimal trajectory for the interval I_2:

$$x^*(t)_{I_2}=\frac{2\hat{d}(t_2-t_3)(1-\rho_2)(t-t_1)}{1-\rho_1}+\hat{d}(t^2-t_1^2)-\hat{a}(t-t_1)-2\hat{d}t_2(t-t_1)+x_1, \quad (38)$$

optimal controls for the interval I_2:

$$u_i^*(t)_{I_2}=-\frac{\hat{d}(t_2-t_3)(1-\rho_2)}{(1-\rho_1)}+\hat{d}t_2-\hat{d}t+a_i. \quad (39)$$

We know that $x^*(t_2)_{I_2}=x_2$, hence

$$x_2=-\hat{a}(t_2-t_1)-\hat{d}(t_1-t_2)^2+\frac{2(1-\rho_2)\hat{d}(t_2-t_3)(t_2-t_1)}{1-\rho_1}+x_1. \quad (40)$$

The value $(1-\rho_1)\int_{t_1}^{t_2}(h_1(x(t),u(t))+h_2(x(t),u(t))dt+\Phi_3(x_2)$ will be considered as the terminal payoff for the interval I_1. Substitute (38), (39), (40) and find the value of the terminal payoff:

$\Phi_2(x_1)=(1-\rho_1)\int_{t_1}^{t_2}(u_1^*(t)_{I_2}(a_1-\frac{1}{2}u_1^*(t)_{I_2})+u_2^*(t)_{I_2}(a_2-\frac{1}{2}u_2^*(t)_{I_2})-d_1x^*(t)_{I_2}-d_2x^*(t)_{I_2})dt=$

$$=(1-\rho_1)\hat{d}x_1(t_1-t_2)+(1-\rho_2)(t_2-t_3)\hat{d}x_1+C(t_1,t_2,t_3), \quad (41)$$

where $C(t_1,t_2,t_3)$ — expression independent of x_1.

Interval I_1.

To find the profile of optimal strategies we have to solve the maximization problem $\int_{t_0}^{t_1}(h_1(x(t),u(t))+h_2(x(t),u(t))dt+\Phi_2(x_1)$ for dynamic (10), initial condition $x(t_0)=x_0$ and terminal payoff (41). By using the Pontryagin's maximum principle we get: optimal controls for the interval I_1:

$$u_i^*(t)_{I_1}=-(t_1-t_2)(1-\rho_1)\hat{d}-(t_2-t_3)(1-\rho_2)\hat{d}+\hat{d}t_1-\hat{d}t+a_i, \quad (42)$$

optimal trajectory for the interval I_1:

$$x_{I_1}^*(t)=2((t_1-t_2)(1-\rho_1)\hat{d}+(t_2-t_3)(1-\rho_2)\hat{d}-\hat{d}t_1)t+\hat{d}t^2-\hat{a}t+x_0. \quad (43)$$

Notice, that $x_{I_1}^*(t_1)=x_1$, hence,

$$x_1=2((t_1-t_2)(1-\rho_1)\hat{d}+(t_2-t_3)(1-\rho_2)\hat{d})t_1-\hat{d}t_1^2-\hat{a}t_1+x_0. \quad (44)$$

Let us substitute (40), (44) in (34), (35), (38), (39) and get expressions for optimal trajectories and controls:

$$x^*(t)_{I_1}=2t\hat{d}(-t_1+(1-\rho_1)(t_1-t_2)+(1-\rho_2)(t_2-t_3))+\hat{d}t^2-\hat{a}t+x_0,$$

$$u_i^*(t)_{I_1} = \hat{d}(t_1 - (1-\rho_1)(t_1 - t_2) - (1-\rho_2)(t_2 - t_3)) - \hat{d}t + a_i, \quad t \in [t_0; t_1],$$

$$x^*(t)_{I_2} = \left(-2\hat{d}t_2 + \frac{2(1-\rho_2)\hat{d}(t_2 - t_3)}{1-\rho_1}\right)(t - t_1) + \hat{d}t^2 - \hat{a}t -$$

$$-2\hat{d}t_1^2 + x_0 + 2t_1(1-\rho_2)\hat{d}(t_2 - t_3) + 2t_1(1-\rho_1)\hat{d}(t_1 - t_2),$$

$$u_i^*(t)_{I_2} = -\frac{\hat{d}(t_2 - t_3)(1-\rho_2)}{(1-\rho_1)} + \hat{d}t_2 - \hat{d}t + a_i, \quad t \in (t_1; t_2],$$

$$x^*(t)_{I_3} = \hat{d}t^2 - \hat{a}t - 2\hat{d}t_3 t + 2\hat{d}t_3 t_2 - 2\hat{d}t_2^2 - 2\hat{d}t_1^2 + 2\hat{d}t_1 t_2 + \frac{2(1-\rho_2)\hat{d}(t_2 - t_3)(t_2 - t_1)}{1-\rho_1} +$$

$$+x_0 + 2t_1(1-\rho_2)\hat{d}(t_2 - t_3) + 2t_1(1-\rho_1)\hat{d}(t_1 - t_2),$$

$$u_i^*(t)_{I_3} = -\hat{d}(t - t_3) + a_i, \quad t \in (t_2; t_3].$$

8. Numeric example

Consider the previous example with numeric parameters.
Let $a_1 = 5$, $a_2 = 6$, $d_1 = 1$, $d_2 = 2$, $p_1 = 0.2$, $p_2 = 0.4$, $\varrho_1 = 0.7$, $\varrho_2 = 0.2$, $\rho_1 = 0.76$, $\rho_2 = 0.96$, $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, $x_0 = 80$.
Consequently:

$$x_{I_1}^*(t) = 3t^2 - 18.92t + 80,$$

$$u_{1_{I_1}}^*(t) = -3t + 8.96,$$

$$u_{2_{I_1}}^*(t) = -3t + 9.96, \quad t \in [t_0; t_1],$$

$$x_{I_2}^*(t) = 3t^2 - 25t + 86.08,$$

$$u_{1_{I_2}}^*(t) = -3t + 12,$$

$$u_{2_{I_2}}^*(t) = -3t + 13, \quad t \in [t_1; t_2],$$

$$x_{I_3}^*(t) = 3t^2 - 35t + 106.08,$$

$$u_{1_{I_3}}^*(t) = -3t + 17,$$

$$u_{2_{I_3}}^*(t) = -3t + 18, \quad t \in [t_2; t_3]$$

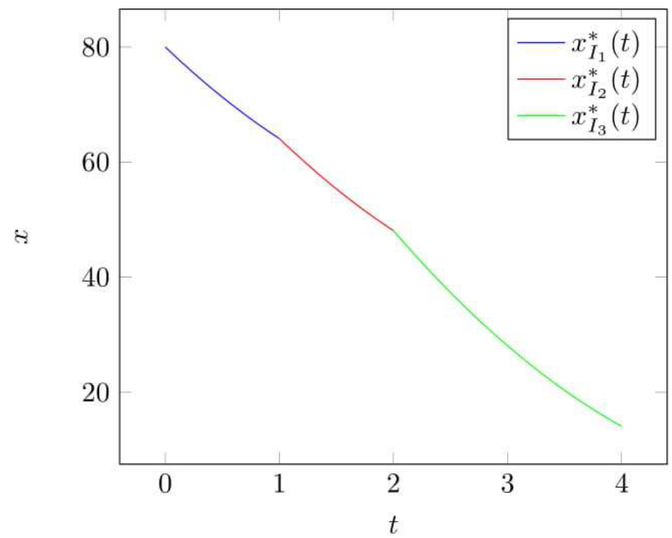

Fig. 9. Optimal trajectory

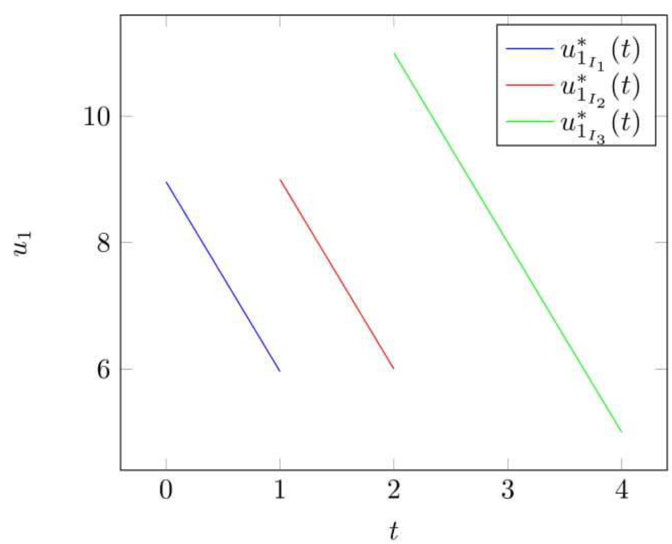

Fig. 10. Optimal control for the player 1

9. Conclusion

In this paper we considered two different approaches to the calculation of optimal controls and trajectory in differential games with random duration. It was constructed a new approach — parametrization, which gives the same answer as the traditional method by using terminal payoff.

References

Afanasyev, V., Kolmanovskiy, V., Nosov, V. (2003). Mathematical theory of designing control systems, P. 617.

Basar, T., Olsder, G. (1995). Dynamic Noncooperative Game Theory. London: Academic Press.

Bellman, R. (1957). Dynamic Programming. Princeton: Princeton University Press.

Breton, M., Zaccour, G., Zahaf, A. (2005). *A differential game of joint implementation of environmental projects*. Automatica, **41(10)**, 1737–1749.

Dockner, E. J., Jorgensen, S., Long, N. V, Sorger, G. (2000). *Differential Games in Economics and Management Science*. Cambridge University Press.

Engwerda, J. (2005). *LQ Dynamic Optimization and Differential Games*. Wiley.

Gromov, D., Gromova, E. (2014). *Differential games with random duration: a hybrid systems formulation*. Contributions to Game Theory and Management, **7**, 104–119.

Gromov, D., Gromova, E. (2017). *On a class of hybrid differential games*. Dynamic games and applications, **7(2)**, 266–288.

Gromova, E. (2016). *The Shapley value as a sustainable cooperative solution in differential games of 3 players*. Recent Advances in Game Theory and Applications, 67–89.

Gromova, E., Tur, A., Balandina, L. (2016). *A game-theoretic model of pollution control with asymmetric time horizons*. Contributions to Game Theory and Management, **9**, 170–179.

Gromova, E., Tur, A. (2017). *On the form of integral payoff in differential games with random duration*. XXVI International Conference on Information, Communication and Automation Technologies (ICAT), IEEE, DOI: 10.1109/ICAT.2017.8171597.

Gromova, E., Malakhova, A., Palestini, A. (2018). *Payoff Distribution in a Multi-Company Extraction Game with Uncertain Duration*. Mathematics, **6(9)**, P. 165.

Gromova, E., Plekhanova, T. (2019). *On the regularization of a cooperative solution in a multistage game with random time horizon*. Discrete Applied Mathematics, **255**, 40–55.

Haurie, A., Krawczyk, J., Zaccour, G. (2012). *Games and dynamic games*. World Scientific Books, control and differential equations, **211**, 370–376.

Jørgensen, S., Zaccour, G. (2007). *Developments in Differential Game Theory and Numerical Methods: Economic and Management Applications*. Computational Management Science, **4(2)**, 159–182.

Kostyunin, S., Shevkoplyas, E. (2011). *On simplification of integral payoff in differential games with random duration*. Vestnik of St.Petersburg Univ., Ser. 10, **4**, 47–56.

Kostyunin, S., Palestini, A., Shevkoplyas, E. (2014). *On a Nonrenewable Resource Extraction Game Played by Asymmetric Firms*. Journal of Optimization Theory and Applications, (2), 660–673.

Petrosjan, L., Danilov, N. (1982). *Cooperative differential games and their applications*, Tomsk University Press.

Petrosjan, L., Murzov, N. (1966). *Game-theoretic problems of mechanics*. Litovsk. Math. Sb., **VI-3**, 423–433. (in Russian)

Petrosjan, L., Shevkoplyas, E. (2000). *Cooperative differential games with random duration*. Vestnik Sankt-Peterburgskogo Universiteta. Ser 1. Matematika Mekhanika Astronomiya (4), 18–23.

Petrosyan, L., Zaccour, G. (2003). *Time-consistent Shapley value allocation of pollution cost reduction*. Journal of Economic Dynamics and Control, **27(3)**, 381–398.

Pontryagin, L. (1961). *Mathematical theory of optimal processes*, 1961. P. 392.

Shevkoplyas, E. (2010). *Stable cooperation in differential games with random duration*. Upravlenie bol. syst., **31-1**, 162–190.