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Networks Structure, Equilibria, and Adjustment Dynamics in Network Games with Nonhomogeneous Players *

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Abstract In this paper, we consider the following problem – what affects the Nash equilibrium amount of investment in knowledge when some agents of the complete graph enter another full one. The solution of this problem will allow us to understand exactly how game agents will behave when deciding whether to enter the other net, what conditions and externalities affect it and how the level of future equilibrium amount of investments in knowledge can be predicted.

Keywords: network; network game; Nash equilibrium; externality; productivity; innovation cluster.

1. Introduction

The processes of globalization, post-industrial development and digitalization of the economy make studying of the role of innovative firms in the world economic development extremely significant. In papers (Alcacer and Chung, 2007; Chung and Alcácer, 2002) mathematical models of the international innovative economy are constructed, on the basis of which the behavior of innovative firms is analyzed. In particular, authors of this article consider an important topic: how do firms realize their investment strategy in the development of knowledge, including outside their own region or country. The behavior of agents is determined by various externalities, which can have a completely different nature. Description of secondary effects is one of the most important directions in network game theory that authors of different articles try to analyze (for example, (Katz and Shapiro, 1985) and (Jaffe, Trajtenberg and Henderson, 1993).

There is also another aspect of the question: how to structure and organize their behavior in the best way in constantly changing economic and social conditions. In (Breschi and Lissoni, 2001), the authors of the article try to take a new look at the system of organizing the actions of agent-innovators. It is important to take into account the impact (externalities) that influence agents by the environment, including other network entities. The article (Cooke, 2001) shows the necessity of creation of regional innovative systems based on clusters. From this follows the relevance of the model description of the process of creating more extensive innovative clusters based on existing ones.

In addition, there is a need to model the process of changing the Nash equilibrium investment values, as well as the search for new internal or angular ones.

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This article continues the study of Nash equilibria and its changes in the process of unification complete graphs. However, this paper contains a number of new elements in comparison with previous studies.

To begin with, we study dynamic behavior of agents, not only by generalizing the simple two-period model of endogenous growth of Romer with the production and externalities of knowledge (as in papers (Matveenko and Korolev, 2015; Matveenko and Korolev, 2017)), but also by using difference equations, which makes possible to consider the transitional dynamics by unification of networks.

This simulation allows us to create a base, which has the potential for its complication and improvement in order to increase the applied nature of the model, for mathematical description of the game behavior of agents.

We show that at different levels of productivity and sizes of the united networks, the equilibrium investment values also change.

Our article considers both internal equilibria and corner ones, which in studies, for the most part, were ignored by researchers.

In this article, we consider the case of unification two complete graphs with some group connection agents, which tend to another network for a number of different reasons: increasing the receiving externalities, the level of productivity, etc.

The content of the article is as follows. Section 2 describes a two-period growth model, characterizes internal Nash equilibrium investments, and analyzes the behavior of network agents. Section 3 defines adjusting dynamics in networks and dynamic stability of equilibria. Section 4 describes the following situation. There are two complete networks. The first network contains nodes. All the agents of this network have the same productivity . The second network contains nodes, with the same agents' productivity . The both networks are initially in inner equilibria. Then in some moment of time any agents of the first network connected to the all agents of the second network. Then transient dynamics occurs and the united network comes in the corner equilibrium, in which the agents are hyperactive. Section 5 summarizes, and lists possible directions for future researches.

2. Model description

There is a network (undirected graph) with n nodes, $i = 1, 2, \ldots, n$; each node represents an agent. In period 1 each agent i possesses initial endowment of good, e, and uses it partially for consumption in first period of life, c_1^i, and partially for investment into knowledge, k_i:

$$c_1^i + k_i = e, \ i = 1, 2, \ldots, n.$$

Investment immediately transforms one-to-one into knowledge which is used in production of good for consumption in second period, c_2^i.

Preferences of agent i are described by quadratic utility function:

$$U_i\left(c_1^i, c_2^i\right) = c_1^i\left(e - ac_1^i\right) + b_i c_2^i,$$

where $b_i > 0$; a is a satiation coefficient, b_i is a parameter, characterized the value of comfort and health in the second period of life compared to consumption in the first period. It is assumed that $c_1^i \in [0, e]$, the utility increases in c_1^i, and is concave (the marginal utility decreases) with respect to c_1^i. These assumptions are equivalent to condition $0 < a < 1/2$.

Production in node i is described by production function:

$$F(k_i, K_i) = B_i k_i K_i,\ B_i > 0$$

which depends on the state of knowledge in i-th node, k_i, and on *environment*, K_i, B_i is a technological coefficient. The environment is the sum of investments by the agent himself and her neighbors:

$$K_i = k_i + \tilde{K}_i, \tilde{K}_i = \sum_{j \in N(i)} k_j,$$

where $N(i)$ – is the set of neighboring nodes of node i.

We will denote the product $b_i B_i$ by A_i and assume that $a < A_i$. Since increase of any of parameters b_i, B_i promotes increase of the second period consumption, we will call A_i “productivity”. We will assume that $A_i \neq 2a, i = 1, 2, ..., n$. If $A_i > 2a$, we will say that i-th agent is *productive*, and if $A_i < 2a$, we will say that i-th agent is *unproductive*.

Three ways of behavior are possible: agent i is called *passive* if she makes zero investment, $k_i = 0$ (i.e. consumes the whole endowment in period 1); *active* if $0 < k_i < e$; *hyperactive* if she makes maximally possible investment e (i.e. consumes nothing in period 1).

Let us consider the following game. Players are the agents $i = 1, 2, \dots, n$. Possible actions (strategies) of player i are values of investment k_i from the segment $[0, e]$. *Nash equilibrium with externalities* (for shortness, *equilibrium*) is a profile of knowledge levels (investments) $(k_1^*, k_2^*, \dots, k_n^*)$, such that each $k_i^*$is a solution of the following problem $P(K_i)$ of maximization of i-th player's utility given environment K_i:

$$U_i\left(c_1^i, c_2^i\right) \underset{c_1^i, c_2^i, k_i}{\longrightarrow} \max$$

$$\begin{cases} c_1^i \le e - k_i, \\ c_2^i \le F(k_i, K_i), \\ c_1^i \ge 0,\ c_2^i \ge 0,\ k_i \ge 0, \end{cases}$$

where the environment K_i is defined by the profile $(k_1^*, k_2^*, \dots, k_n^*)$:

$$K_i = k_i^* + \sum_{j \in N(i)} k_j^*$$

The first two constraints of problem $P(K_i)$ in the optimum point are evidently satisfied as equalities. Substituting into the objective function, we obtain a new function (*payoff function*):

$$V_i(k_i, K_i) = U_i\big(e - k_i, F_i(k_i, K_i)\big) = (e - k_i)\big(e - a(e - K_i)\big) + A_i k_i K_i =$$

$$= e^2(1 - a) - k_i e(1 - 2a) - a k_i^2 + A_i k_i K_i. \quad (1)$$

If all players' solutions are internal, i.e. all players are active, the equilibrium will be referred as *inner* equilibrium else it be referred as *corner* equilibrium. Clearly, the inner equilibrium (if it exists for given values of parameters) is defined by the system

$$D_1 V_i(k_i, K_i) = 0,\ i = 1, 2, \dots, n, \quad (2)$$

or

$$D_1V_i\left(k_i, K_i\right) = e\left(2a-1\right) - 2ak_i + A_iK_i = 0,\ \ i = 1, 2, \ldots, n. \tag{3}$$

Let us introduce the following notations: $\tilde{A}$ - diagonal matrix, which has numbers $A_1, A_2, \ldots, A_n$ on the main diagonal, I – unit $n \times n$ matrix, M – network adjacency matrix. In this matrix $M_{ij} = M_{ji} = 1$, if there is the edge of the matrix, connecting vertexes i and j, and $M_{ij} = M_{ji} = -$ otherwise. It is believed, that $M_{ii} = 0$ for all $i = 1, 2, \ldots, n$. The system of equations (3) takes the form:

$$\left(\tilde{A} - 2aI\right)k + \tilde{A}M\,k = \bar{e}, \tag{4}$$

where $k = \left(k_1, k_2, \ldots, k_n\right)^T$, $\bar{e} = \big(e\left(1-2a\right), e\left(1-2a\right), \ldots, e\left(1-2a\right)\big)^T$.

Theorem 1. *(Matveenko, Korolev and Zhdanova, 2017, Theorem 1.1). The system of equations* (4) *for a complete network with homogeneous agents has only decision.*

We introduce the following notation. Regardless of agent type of behavior the equation root

$$D_1V_i\left(k_i, K_i\right) = \left(A_i - 2a\right)k_i + A_i\tilde{K}_i - e\left(1-2a\right) = 0$$

will be denoted by $\tilde{k}_i^s$*. Thus,*

$$\tilde{k}_i^s = \frac{e(2a-1) + A_i\tilde{K}_i}{2a - A_i},$$

where $\tilde{K}_i$ *– pure externality of agent* i*. It is obvious, that if agent* i *is active, then his investments will be equal to* $\tilde{k}_i^s$ *in equilibrium. To analyze equilibriums we need the following statement.*

Proposition 1. *(Matveenko, Korolev and Zhdanova, 2017, Lemma 2.1 and Corollary 2.1) A set of investment agent values* $(k_1, k_2, \ldots, k_n)$ *can be an equilibrium only if for each* $i = 1, 2, \ldots, n$ *it is true that*

1. *if* $k_i = 0$*, then* $\tilde{K}_i \leq \frac{e(1-2a)}{A_i}$*;*
2. *if* $0 < k_i < e$*, then* $k_i = \tilde{k}_i^S$*;*
3. *if* $k_i = e$*, then* $\tilde{K}_i \geq \frac{e(1-A_i)}{A_i}$*.*

Lemma 1. *(Matveenko, Korolev and Zhdanova, 2017, Lemma 2.2) In equilibrium, the agent* i *is passive if and only if*

$$K_i \leq \frac{e\left(1-2a\right)}{A_i};$$

the agent i *is active if and only if*

$$\frac{e\left(1-2a\right)}{A_i} < K_i < \frac{e}{A_i};$$

the agent i *is hyperactive if and only if*

$$K_i \geq \frac{e}{A_i}.$$

In complete network there is the same environment for all agents, so we get the following consequence.

Corollary 1. *In equilibrium in a complete network, agents with the same productivity make the same investments. If all agents in the full network have the same productivity, then there is homophilia, that is, all agents behave identically.*

Remark 1. In a complete network, there cannot be situations when in equilibrium an agent with a high production value is active or passive, and an agent with a lower productivity is hyperactive, or when an agent with a higher production value is passive, and an agent with a lower productivity is active or hyperactive.

Speaking of a complete network, we will omit the index in the designation of the agent environment, since in a complete network the environment for all agents is the same. Thus, K is denoted the amount of investment of all agents of the complete network.

Corollary 2. *(Matveenko, Korolev and Zhdanova, 2017, Corollary 2.3) The equilibrium, in which all agents are passive, is possible in a complete network if and only if*

$$K \leq \frac{e\,(1-2a)}{\max\limits_i A_i}.$$

The equilibrium, in which all agents are active, is possible in a complete network if and only if

$$\frac{e\,(1-2a)}{\min\limits_i A_i} < K < \frac{e}{\max\limits_i A_i}.$$

The equilibrium, in which all agents are hyperactive, is possible in a complete network if and only if

$$K \geq \frac{e}{\min\limits_i A_i}.$$

Corollary 3. *Equilibrium, in which all agents are hyperactive, is possible in a complete network, if and only if*

$$\min_i A_i \geq \frac{1}{n}.$$

Equilibrium, in which all agents are passive, is always possible.

3. Adjusting dynamics in networks and dynamic stability of equilibria

We introduce adjustment dynamics which may start after a small deviation from equilibrium or after junction of networks each of which was initially in equilibrium. We model the adjustment dynamics in the following way.

Definition 1. Each agent maximizes her utility by choosing a level of investment; at the moment of decision-making she considers her environment as exogenously given. Correspondingly, if $k_i^t = 0$ and $D_1 V_i(k_i, K_i)|_{k_i=0} \leq 0$, then $k_i^{t+1} = 0$, and if $k_i^t = e$ and $D_1\, V_i\,(k_i, K_i)|_{k_i=e} \geq 0$, then $k_i^{t+1} = e$; in all other cases, k_i^{t+1} solves the difference equation:

$$-2ak_i^{t+1} + A_i K_i^t - e\,(1-2a) = 0,\ t = 0, 1, 2, \ldots.$$

Definition 2. The equilibrium is called *dynamically stable* if, after a small deviation of one of the agents from the equilibrium, dynamics starts which returns the equilibrium back to the initial state. In the opposite case the equilibrium is called *dynamically unstable*.

Lemma 2. *In any network, a corner equilibrium, in which all the agents are passive, is stable.*

Proof. As follows from (3), for any agent i the inequality $D_1V_i\left(k_i, K_i\right) \leq 0$ holds as strict: $D_1V_i\left(k_i, K_i\right) = e\left(2a-1\right) < 0$. □

Lemma 3. *In any network, an inner equilibrium is unstable.*

Proof. Let the network consist of n nodes, and all agents have the same productivity A. Let M be the $n \times n$-matrix of difference equations system , describing the dynamics in the network. Then the matrix M has the form $M = \frac{A}{2a}\left(\mathrm{A}+I\right)$, where A – the adjacent matrix of the network, I – identity $n \times n$-matrix. In this way, the matrix M is symmetric and therefore has orthogonal basis of eigenvectors. Let $v_1, v_2, \ldots, v_n$ be an orthogonal basis, and x be arbitrary real n-dimensional vector. We expand it in terms of the basis:

$$x = \sum_{i=1}^{n} c_i v_i .$$

Then:

$$\frac{x^T M x}{x^T x} = \frac{\sum_{i=1}^{n} c_i v_i^T M \sum_{j=1}^{n} c_j v_j}{\sum_{i=1}^{n} c_i v_i^T \sum_{j=1}^{n} c_j v_j} = \frac{\sum_{i,j=1}^{n} c_i c_j \lambda_j v_i^T v_j}{\sum_{i,j=1}^{n} c_i c_j v_i^T v_j} = \\ = \frac{\sum_{j=1}^{n} c_j^2 \lambda_j}{\sum_{j=1}^{n} c_j^2} \leq \frac{\sum_{j=1}^{n} c_j^2 \lambda_1}{\sum_{j=1}^{n} c_j^2} = \lambda_1, \quad (5)$$

where λ_1 means the greatest eigenvalue. Choosing as a x vector, all components of which are equal to 1, we get:

$$\lambda_1 \geq \frac{A}{2a} \cdot \frac{2m+n}{n}, \quad (6)$$

where m means the number of edges in network, and n means the number of nodes.

Let h be the number of node, having the greatest degree $d_{\max}$. Consider the vector x, the components of which are given in the following way:

$$x_i = \begin{cases} \sqrt{d_{\max}}, & if \quad i = h, \\ 1, & if \quad \mathrm{A}_{ih} = 1, \\ 0, & else. \end{cases}$$

Then

$$\sum_{j=1}^{n} M_{ij} x_j \geq \frac{A}{2a} \left\{ \begin{array}{ll} \sqrt{d_{\max}} + d_{\max}, & if \quad i = h, \\ \sqrt{d_{\max}} + 1, & if \quad M_{ih} = 1, \\ 0, & else \end{array} \right\} = \frac{A}{2a}\left(\sqrt{d_{\max}}+1\right) x_i. \quad (7)$$

Multiplying both sides of (7) by x_i and summing over i, we obtain the inequality

$$x^T M x \geq \frac{A}{2a}\left(\sqrt{d_{\max}}+1\right) x^T x,$$

and, applying (5), we get

$$\lambda_1 \geq \frac{x^T M x}{x^T x} \geq \frac{A}{2a}\left(\sqrt{d_{\max}} + 1\right). \tag{8}$$

Similarly to (5), we get

$$\frac{x^T M x}{x^T x} = \frac{\sum_{i=1}^{n} c_i v_i^T M \sum_{j=1}^{n} c_j v_j}{\sum_{i=1}^{n} c_i v_i^T \sum_{j=1}^{n} c_j v_j} = \frac{\sum_{i,j=1}^{n} c_i c_j \lambda_j v_i^T v_j}{\sum_{i,j=1}^{n} c_i c_j v_i^T v_j} = \\ = \frac{\sum_{j=1}^{n} c_j^2 \lambda_j}{\sum_{j=1}^{n} c_j^2} \geq \frac{\sum_{j=1}^{n} c_j^2 \lambda_n}{\sum_{j=1}^{n} c_j^2} = \lambda_n, \tag{9}$$

where λ_n means the least eigenvalue of matrix M. We now consider the vector x, which components are given as follows:

$$x_i = \begin{cases} -\sqrt{d_{\max}}, & if \quad i = h, \\ -1, & if \quad \mathrm{A}_{ih} = 1, \\ 0, & otherwise. \end{cases}$$

Then

$$\sum_{j=1}^{n} M_{ij} x_j \leq \frac{A}{2a} \left\{ \begin{array}{ll} -\sqrt{d_{\max}} - d_{\max}, & if \quad i = h, \\ -\sqrt{d_{\max}} - 1, & if \quad M_{ih} = 1, \\ 0, & otherwise \end{array} \right\} = \frac{A}{2a}\left(-\sqrt{d_{\max}} - 1\right) x_i. \tag{10}$$

Multiplying both sides of (10) by x_i and summing over i, we obtain the inequality

$$x^T M x \leq \frac{A}{2a}\left(-\sqrt{d_{\max}} - 1\right) x^T x,$$

and, applying (9), we get

$$\lambda_n \leq \frac{x^T M x}{x^T x} \leq -\frac{A}{2a}\left(\sqrt{d_{\max}} + 1\right). \tag{11}$$

So, in the case when all agents have the same productivity and there is at least one edge in the network, the largest eigenvalue of the equation system, describing the dynamics, according to (8), is always greater than unity. If the network does not have any edges (all agents are isolated), then an internal equilibrium is possible only if all agents are productive
($A > 2a$), but then according to (8) the largest eigenvalue is greater then unit. □

4. Network dynamics model of net unification

Let us consider the following situation. There are two complete networks with $n_1 + n_3$ nodes with productivities A_1 and with n_2 nodes with productivities A_2. Let's pretend that n_3 agents of the first network decide to connect with agents of the second network. So there are three types of agents in the united network. The actors of the first type are all the agents of the first network, besides the agents of the first network, which connected to the agents of the second network. Actors of the second type are all agents of the second network. The third type of actors is n_3 agents of the first network that connected to all actors of the second network. Since all agents of the same type will have the same environment, they will behave in the same way, not only in equilibrium, but also in dynamics. Therefore, the investment of each agent of the type i will be denoted k_i, and the environment of each agent of the type i will be denoted K_i.

Both the complete networks are initially in inner equilibrium. It follows immediately from (3) that the initial investment of agents is the following

$$k_1^0 = k_3^0 = \frac{e(1-2a)}{(n_1+n_3)A_1 - 2a}, \tag{12}$$

$$k_2^0 = k_3^0 = \frac{e(1-2a)}{n_2A_2 - 2a}. \tag{13}$$

The system (3) for inner equilibrium in joined network is

$$\begin{cases} (n_1A_1 - 2a)k_1 + A_1n_3k_3 = e(1-2a), \\ (n_2A_2 - 2a)k_2 + A_2n_3k_3 = e(1-2a), \\ n_1A_1k_1 + n_2A_1k_2 + (n_3A_1 - 2a)k_3 = e(1-2a). \end{cases} \tag{14}$$

Definition 1 implies that the dynamics in model under consideration is described by the system of differential equations:

$$\begin{cases} k_1^{t+1} = \frac{n_1A_1}{2a}k_1^t + \frac{n_3A_1}{2a}k_3^t + \frac{e(2a-1)}{2a}, \\ k_2^{t+1} = \frac{n_2A_2}{2a}k_2^t + \frac{n_3A_2}{2a}k_3^t + \frac{e(2a-1)}{2a}, \; where\ t = 0, 1, 2\ldots \\ k_3^{t+1} = \frac{n_1A_1}{2a}k_1^t + \frac{n_2A_1}{2a}k_2^t + \frac{n_3A_1}{2a}k_3^t + \frac{e(2a-1)}{2a}. \end{cases} \tag{15}$$

Characteristic equation for this system is

$$\begin{vmatrix} \frac{n_1A_1}{2a} - \lambda & 0 & \frac{n_3A_1}{2a} \\ 0 & \frac{n_2A_2}{2a} - \lambda & \frac{n_3A_2}{2a} \\ \frac{n_1A_1}{2a} & \frac{n_2A_1}{2a} - \lambda & \frac{n_3A_1}{2a} - \lambda \end{vmatrix} =$$

$$= \left(\frac{n_1A_1}{2a} - \lambda\right)\left[\lambda^2 - 2\frac{n_3A_1 + n_2A_2}{2a}\lambda - \frac{n_2A_2n_3A_1}{4a^2}\right] = 0 \tag{16}$$

To find an explicit solution of a system of difference equations (3.1) we need to impose the restrictions

$$n_1 = n_3, \; n_1A_1 = n_2A_2, \tag{17}$$

i.e. all the three nets have the same total productivity.

Then the system (15) takes the form

$$\begin{cases} k_1^{t+1} = \frac{n_1A_1}{2a}k_1^t + \frac{n_1A_1}{2a}k_3^t + \frac{e(2a-1)}{2a}, \\ k_2^{t+1} = \frac{n_1A_1}{2a}k_2^t + \frac{n_1A_2}{2a}k_3^t + \frac{e(2a-1)}{2a}, \\ k_3^{t+1} = \frac{n_1A_1}{2a}k_1^t + \frac{n_2A_1}{2a}k_2^t + \frac{n_1A_1}{2a}k_3^t + \frac{e(2a-1)}{2a}. \end{cases} \tag{18}$$

and the equation (16) takes the form

$$\left(\frac{n_1A_1}{2a} - \lambda\right)\left(\lambda^2 - 2\frac{n_1A_1}{2a}\lambda - \left(\frac{n_1A_1}{2a}\right)^2\right) = 0,$$

hence

$$\lambda_1 = \frac{n_1 A_1}{2a},$$
$$\lambda_{2,3} = \frac{n_1 A_1}{2a} \pm \sqrt{2}\frac{n1 A_1}{2a}.$$

Let us find the eigenvectors. For the eigenvalue $\lambda_1 = \frac{n_1 A_1}{2a}$, we obtain the system of equations

$$\begin{pmatrix} 0 & 0 & \frac{n_1 A_1}{2a} \\ 0 & 0 & \frac{n_1 A_2}{2a} \\ \frac{n_1 A_1}{2a} & \frac{n_2 A_1}{2a} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

thus as the first eigenvector we can take

$$e_1 = \begin{pmatrix} n_2 \\ -n_1 \\ 0 \end{pmatrix}.$$

For the second eigenvalue $\lambda_2 = (1 + \sqrt{2}\frac{n_1 A_1}{2a})$ we obtain the system

$$\begin{pmatrix} -\sqrt{2}\frac{n_1 A_1}{2a} & 0 & \frac{n_1 A_1}{2a} \\ 0 & -\sqrt{2}\frac{n_1 A_1}{2a} & \frac{n_1 A_2}{2a} \\ \frac{n_1 A_1}{2a} & \frac{n_2 A_1}{2a} & -\sqrt{2}\frac{n_1 A_1}{2a} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or in view of (17),

$$\begin{cases} -\sqrt{2}\frac{n_1 A_1}{2a} x_1 + \frac{n_1 A_1}{2a} x_3 = 0, \\ -\sqrt{2}\frac{n_1 A_1}{2a} x_2 + \frac{n_1 A_2}{2a} x_3 = 0, \\ \frac{n_1 A_1}{2a} x_1 + \frac{n_2 A_1}{2a} x_2 - \sqrt{2}\frac{n_1 A_1}{2a} x_3 = 0, \end{cases} \tag{19}$$

thus as the second eigenvector we can take

$$e_2 = \begin{pmatrix} \frac{A_1}{\sqrt{2}} \\ \frac{A_2}{\sqrt{2}} \\ A_1 \end{pmatrix}.$$

For the third eigenvalue $\lambda_3 = (1 - \sqrt{2})\frac{n_1 A_1}{2a}$ we obtain the system

$$\begin{pmatrix} \sqrt{2}\frac{n_1 A_1}{2a} & 0 & \frac{n_1 A_1}{2a} \\ 0 & \sqrt{2}\frac{n_1 A_1}{2a} & \frac{n_1 A_2}{2a} \\ \frac{n_1 A_1}{2a} & \frac{n_2 A_1}{2a} & \sqrt{2}\frac{n_1 A_1}{2a} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or in view of (17),

$$\begin{cases} \sqrt{2}\frac{n_1 A_1}{2a} x_1 + \frac{n_1 A_1}{2a} x_3 = 0, \\ \sqrt{2}\frac{n_1 A_1}{2a} x_2 + \frac{n_1 A_2}{2a} x_3 = 0, \\ \frac{n_1 A_1}{2a} x_1 + \frac{n_2 A_1}{2a} x_2 + \sqrt{2}\frac{n_1 A_1}{2a} x_3 = 0, \end{cases}$$

thus as the third eigenvector we can take

$$e_2 = \begin{pmatrix} -\frac{A_1}{\sqrt{2}} \\ -\frac{A_2}{\sqrt{2}} \\ A_1 \end{pmatrix}.$$

Thus, dynamics in joined network is described with following vector equation

$$\begin{pmatrix} k_1^t \\ k_2^t \\ k_3^t \end{pmatrix} = C_1 \left(\frac{n_1 A_1}{2a} \right)^t \begin{pmatrix} n_2 \\ -n_1 \\ 0 \end{pmatrix} + C_2 \left(\left(1 + \sqrt{2}\right) \frac{n_1 A_1}{2a} \right)^t \begin{pmatrix} \frac{A_1}{\sqrt{2}} \\ \frac{A_2}{\sqrt{2}} \\ A_1. \end{pmatrix} +$$

$$+ C_3 \left(\left(1 + \sqrt{2}\right) \frac{n_1 A_1}{2a} \right)^t \begin{pmatrix} -\frac{A_1}{\sqrt{2}} \\ -\frac{A_2}{\sqrt{2}} \\ A_1. \end{pmatrix} + \begin{pmatrix} k_1^* \\ k_2^* \\ k_3^* \end{pmatrix}, \quad t = 0, 1, 2, \ldots. \quad (20)$$

The constants C_1, C_2, C_3 can be found from initial conditions. Before unification the both networks were in symmetric inner equilibria (12)-(13). Thus, the initial conditions, taking into account (17), are

$$k_1^0 = k_3^0 = \frac{e(1-2a)}{2n_1 A_1 - 2a}, \; k_2^0 = \frac{e(1-2a)}{n_1 A_1 - 2a} = \frac{e(1-2a)}{n_2 A_2 - 2a}.$$

The system (14) in view of (17) takes the form

$$\begin{cases} (n_1 A_1 - 2a) k_1 + n_1 A_1 k_3 = e(1-2a), \\ (n_1 A_1 - 2a) k_2 + n_1 A_2 k_3 = e(1-2a), \\ n_1 A_1 k_1 + n_2 A_1 k_2 + (n_1 A_1 - 2a) k_3 = e(1-2a). \end{cases}$$

Solving this system by Kramer formulas, we obtain

$$k_1^* = \frac{e(1-2a)\left[4a^2 - 2a n_1 A_1 + n_1 n_2 A_1 (A_1 - A_2)\right]}{(n_1 A_1 - 2a)^3},$$

$$k_2^* = \frac{e(1-2a)\left(4a^2 - 4a n_1 A_1 + 2a n_1 A_2)\right)}{(n_1 A_1 - 2a)^3},$$

$$k_2^* = \frac{-e(1-2a)(n_1 A_1 - 2a)(n_2 A_1 + 2a))}{(n_1 A_1 - 2a)^3},$$

Hence by $t = 0$ we receive the following equations

$$\begin{cases} C_1 n_2 + C_2 \frac{A_1}{\sqrt{2}} - C_3 \frac{A_1}{\sqrt{2}} + k_1^* = k_1^0, \\ -C_1 n_1 + C_2 \frac{A_2}{\sqrt{2}} - C_3 \frac{A_2}{\sqrt{2}} + k_2^* = k_2^0, \\ C_2 A_1 + C_3 A_1 + k_3^* = k_3^0. \end{cases}$$

Multiplying the first equation of the system by n_1 and the second equation by n_2 and adding two first equations, we obtain

$$\begin{cases} C_2 (n_1 A_1 + n_2 A_2) - C_3 (n_1 A_1 + n_2 A_2) = \left(k_1^0 - k_1^*\right) n_1 \sqrt{2} + \left(k_2^0 - k_2^*\right) n_2 \sqrt{2} \\ C_2 A_1 + C_3 A_1 = k_3^0 - k_3^* \end{cases}$$

Thus we have

$$C_2 \frac{\left(k_1^0 - k_1^*\right) n_1 \sqrt{2} + \left(k_2^0 - k_2^*\right) n_2 \sqrt{2}}{2(n_1 A_1 + n_2 A_2)} + \frac{k_3^0 - k_3^*}{2A_1},$$

or in view of (17),

$$C_2 \frac{\left(k_1^0 - k_1^*\right) n_1 \sqrt{2} + \left(k_2^0 - k_2^*\right) n_2 \sqrt{2} + 2 n_1 \left(k_3^0 - k_3^*\right)}{4 n_1 A_1}.$$

Thus at reasonable parameters the constant at the largest positive eigenvalue is also positive. Hence the transition process will pass to corner equilibrium, where all the agents are hyperactive.

Let us check that the corner solution, where $k_1 = k_2 = k_3 = e$, is stable equilibrium:

$$D_1 V_1(k_1, K_1)|_{k_1=k_2=k_3=e} = e(2a-1) - 2ae + (n_1+n_3)A_1 e = A_1(n_1+n_3)e - e \geq 0,$$

iff $A_1 \geq \frac{1}{n_1+n_3}$,

$$D_1 V_2(k_2, K_2)|_{k_1=k_2=k_3=e} = e(2a-1) - 2ae + (n_2+n_3)A_2 e = A_2(n_2+n_3)e - e \geq 0,$$

iff $A_2 \geq \frac{1}{n_2+n_3}$,

$$D_1 V_3(k_3, K_3)|_{k_1=k_2=k_3=e} = e(2a-1) - 2ae + (n_1+n_2+n_3)A_1 e = A_1(n_1+n_2+n_3)e - e \geq 0,$$

iff $A_1 \geq \frac{1}{n_1+n_2+n_3}$, corresponding to Corollary 2.4 in (Matveenko, Korolev and Zhdanova, 2017).

Thus, if $A_1 \geq \frac{1}{n_1+n_3}$ and $A_2 \geq \frac{1}{n_2+n_3}$, then the state $k_1 = k_2 = k_3 = e$ is a corner equilibrium. Besides if $A_1 \geq \frac{1}{n_1+n_3}$ and $A_2 \geq \frac{1}{n_2+n_3}$, then this equilibrium is dynamically stable according to definition 2. The transient rate is directly proportional to total productivity of nets, as seen from (20).

5. Conclusion

In this paper, we have described the process of change in game equilibrium during graphs unification using two-stage and dynamics models. We have highlighted the significance of the productivity role that influence the agents' behavior. Moreover, we determined the importance of graphs sizes, which also effect on agents' decisions that they take during unification process.

We believe that this article offers the base model of game equilibria change that can be improved with increasing the amount of parameters and modification graphs' type to incomplete nets or non-oriented graphs.

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