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A Note on Four-Players Triple Game

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Abstract We introduce so-called four-players triple game and define Nash equilibrium. The problem of numerical finding of a Nash equilibrium in a four-players triple game has been examined. Such a game can be completely described by twelve matrices, and it turns out to be equivalent to the solving a nonconvex optimization problem. Special methods of local and global search for the optimization problem are proposed. The proposed algorithm was implemented on test problems by "GAMUT" (http: gamut.stanford.edu).
Keywords: nonconvex optimization, four-players triple game, local and global search algorithm, Nash equilibrium,

1. Introduction

Game theory plays an important role in applied mathematics, mathematical modeling, economics and decision theory. There are many works devoted to game theory (Neumann and Morgenstern, 1944; Vorobyev, 1984; Howson, 1972; Strekalovsky and Orlov, 2007) and (Owen, 1971; Gibbons, 1992; Mangasarian and Stone, 1964). Most of them deals with zero sum two person games or nonzero sum two person games. Also, two person non zero sum game was studied in (Strekalovsky and Orlov, 2007; Strekalovsky and Enkhbat, 2014; Orlov et al., 2014) by reducing it to D.C programming.

The problem of numerical finding of a Nash equilibrium in a 3-player polymatrix game was studied in (Strekalovsky and Enkhbat, 2014; Orlov et al., 2014). In this paper it has found that a game can be completely described by six matrices, and it turns out to be equivalent to the solving a nonconvex optimization problem with a bilinear structure in the objective function.

We consider the four-person matrix game where each of them plays with other three players. We call such game four-players triple game. In this game we introduce a definition of Nash equilibrium similarly to (Strekalovsky and Enkhbat, 2014). The game reduces to a nonconvex optimization problem. For solving the optmization problem, we propose a global optimization method that combines the ideas of the classical multistart and local search methods.

2. Problem formulation and optimality conditions

The four-players triple game is given by $\Gamma_4 = \{i, j, k, t, a, b, c, d\}$, where $i = 1, ..., m$, $j = 1, ..., n, \quad k = 1, ..., s, \quad \ell = 1, ..., p$ are the sets of pure strategies of the respective players 1, 2, 3 and 4.

Payoffs of players defined on a strategy $(i, j, k, t) \in I \times J \times K \times T$ are given by:

$$a(i,j,k,\ell) = a^1_{ijk} + a^2_{ij\ell} + a^3_{ik\ell}, \quad b(i,j,k,\ell) = b^1_{ijk} + b^2_{ij\ell} + b^3_{jk\ell},$$

$$c(i,j,k,\ell) = c^1_{ijk} + c^2_{ik\ell} + c^3_{jk\ell}, \quad d(i,j,k,\ell) = d^1_{ij\ell} + d^2_{jk\ell} + c^3_{ik\ell},$$

$$i = 1, \dots, m, \quad j = 1, \dots, n, \quad k = 1, \dots, s, \quad \ell = 1, \dots, p.$$

Thus, twelve matrices are given for players A, B, C, D, where $A = (A_1, A_2, A_3)$, $B = (B_1, B_2, B_3), \quad C = (C_1, C_2, C_3)$ and $D = (D_1, D_2, D_3)$.

The payoff functions of the first and second players are defined as:

$$F_1(x,y,z,t) = \sum_{i=1}^{m} x_i \left(\sum_{j=1}^{n} \sum_{k=1}^{s} a^1_{ijk} y_j z_k + \sum_{j=1}^{n} \sum_{\ell=1}^{p} a^2_{ij\ell} y_j t_\ell + \sum_{k=1}^{s} \sum_{\ell=1}^{p} a^3_{ik\ell} z_k t_\ell \right),$$

$$F_2(x,y,z,t) = \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} \sum_{k=1}^{s} b^1_{ijk} x_i z_k + \sum_{i=1}^{m} \sum_{\ell=1}^{p} b^2_{ij\ell} x_i t_\ell + \sum_{k=1}^{s} \sum_{\ell=1}^{p} b^3_{ik\ell} z_k t_\ell \right),$$

where (x, y, z, t) vector of mixed strategies of four players. Similarly, we can define the payoff functions F_3 and F_4 of other players.

Denote by S_q the set $S_q = \{u \in \mathbb{R}^q | \sum_{\tau=1}^{q} u_i = 1, \ u_i \geq 0, \ \tau = 1, \dots, q\}, \quad q = m, n, s, p$.

Definition 1. A strategy $(x^*, y^*, z^*, t^*) \in S_m \times S_n \times S_s \times S_p$ is called a Nash equilibrium of the four-person matrix game if the following conditions are satisfied:

$$\begin{array}{ll} F_1(x^*,y^*,z^*,t^*) \geq F_1(x,y^*,z^*,t^*), & \forall x \in S_m, \\ F_2(x^*,y^*,z^*,t^*) \geq F_2(x^*,y,z^*,t^*), & \forall y \in S_n, \\ F_3(x^*,y^*,z^*,t^*) \geq F_3(x^*,y^*,z,t^*), & \forall z \in S_s, \\ F_4(x^*,y^*,z^*,t^*) \geq F_4(x^*,y^*,z^*,t), & \forall t \in S_p. \end{array} \tag{1}$$

Now we formulate next theorem:

Theorem 1. *The stratergy $(x^*,y^*,z^*,t^*) \in S_m \times S_n \times S_\ell \times S_p$ is a Nash equilibrium in the game $\Gamma_4 = \imath(A,B,C,D)$ if and only if there exist numbers $\alpha_*, \beta_*, \gamma_*, \delta_*$ such that:*

$$\begin{array}{l} \sum_{j=1}^{n}\sum_{k=1}^{s} a^1_{ijk} y^*_j z^*_k + \sum_{j=1}^{n}\sum_{\ell=1}^{p} a^2_{ij\ell} y^*_j t^*_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} a^3_{ik\ell} z^*_k t^*_\ell \leq \alpha_* e_m \\ \sum_{i=1}^{m}\sum_{k=1}^{s} b^1_{ijk} x^*_i z^*_k + \sum_{i=1}^{m}\sum_{\ell=1}^{p} b^2_{ij\ell} x^*_i t^*_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} b^3_{jk\ell} z^*_k t^*_\ell \leq \beta_* e_n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} c^1_{ijk} x^*_i y^*_j + \sum_{i=1}^{m}\sum_{\ell=1}^{p} c^2_{ik\ell} x^*_i t^*_\ell + \sum_{j=1}^{n}\sum_{\ell=1}^{p} c^3_{jk\ell} y^*_j t^*_\ell \leq \gamma_* e_s \\ \sum_{i=1}^{m}\sum_{j=1}^{n} d^1_{ij\ell} x^*_i y^*_j + \sum_{i=1}^{m}\sum_{k=1}^{s} d^2_{ik\ell} x^*_i z^*_k + \sum_{j=1}^{n}\sum_{k=1}^{s} d^3_{jk\ell} y^*_j z^*_k \leq \delta_* e_p \end{array} \tag{2}$$

and satisfy

$$F_1(x^*,y^*,z^*,t^*) + F_2(x^*,y^*,z^*,t^*) + F_3(x^*,y^*,z^*,t^*) + F_4(x^*,y^*,z^*,t^*) = \\ = \alpha_* + \beta_* + \gamma_* + \delta_*. \tag{3}$$

Proof. Necessity: Assume that (x^*,y^*,z^*,t^*) is a Nash equilibrium. Then by definition 1, we have

$$F_1(x^*,y^*,z^*,t^*) \geq F_1(x,y^*,z^*,t^*), \quad \forall x \in S_m, \tag{4}$$

$$F_2(x^*,y^*,z^*,t^*) \geq F_2(x^*,y,z^*,t^*), \quad \forall y \in S_n, \tag{5}$$

$$F_3(x^*,y^*,z^*,t^*) \geq F_3(x^*,y^*,z,t^*), \quad \forall z \in S_s, \tag{6}$$

$$F_4(x^*,y^*,z^*,t^*) \geq F_4(x^*,y^*,z^*,t), \quad \forall t \in S_p. \tag{7}$$

In the first inequality (4), successively choose $x = (0,0,\ldots,1,\ldots,0)$ with 1 in each of the m spots, in (5) choose $y = (0,0,\ldots,1,\ldots,0)$ with 1 in each of the n spots, in (6) choose $z = (0,0,\ldots,1,\ldots,0)$ with 1 in each of the s spots and in (7) choose $t = (0,0,\ldots,1,\ldots,0)$ with 1 in each of the p spots. We can easily see that

$$F_1(x^*,y^*,z^*,t^*) \geq \left(\sum_{j=1}^{n}\sum_{k=1}^{s} a^1_{ijk} y^*_j z^*_k + \sum_{j=1}^{n}\sum_{\ell=1}^{p} a^2_{ij\ell} y^*_j t^*_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} a^3_{ik\ell} z^*_k t^*_\ell \right),$$

$$i = 1,\ldots,m,$$

Choose scalar $\alpha^* = F_1(x^*,y^*,z^*,t^*)$ and summing these inequalities, we have inequality (4). For other players doing the same procedure we obtain (2).

Sufficiency: Suppose that for a vector
$(x^*, y^*, z^*, t^*) \in S_m \times S_n \times S_\ell \times S_p$ and numbers $\alpha_*, \beta_*, \gamma_*, \delta_*$ conditions (2) and (3) are satisfied.
We choose $x \in S_m,\ y \in S_n,\ z \in S_s$ and $t \in S_p$ and multiply (4)-(7) by x, y, z and t respectively. In summing we obtain

$$\left(\sum_{i=1}^{m} x_i\right) F_1(x^*, y^*, z^*, t^*) \geq$$
$$\sum_{i=1}^{m} x_i \left(\sum_{j=1}^{n}\sum_{k=1}^{s} a^1_{ijk} y^*_j z^*_k + \sum_{j=1}^{n}\sum_{\ell=1}^{p} a^2_{ij\ell} y^*_j t^*_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} a^3_{ik\ell} z^*_k t^*_\ell\right),$$

$$\left(\sum_{j=1}^{n} y_j\right) F_2(x^*, y^*, z^*, t^*) \geq$$
$$\sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m}\sum_{k=1}^{s} b^1_{ijk} x^*_i z^*_k + \sum_{i=1}^{m}\sum_{\ell=1}^{p} b^2_{ij\ell} x^*_i t^*_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} b^3_{ik\ell} z^*_k t^*_\ell\right),$$

$$\left(\sum_{k=1}^{s} z_k\right) F_3(x^*, y^*, z^*, t^*) \geq$$
$$\sum_{k=1}^{s} z_k \left(\sum_{i=1}^{m}\sum_{j=1}^{n} c^1_{ijk} x^*_i y^*_j + \sum_{i=1}^{m}\sum_{\ell=1}^{p} c^2_{ik\ell} x^*_i t^*_\ell + \sum_{j=1}^{n}\sum_{\ell=1}^{p} c^3_{jk\ell} y^*_j t^*_\ell\right),$$

$$\left(\sum_{\ell=1}^{p} y_j\right) F_4(x^*, y^*, z^*, t^*) \geq$$
$$\sum_{\ell=1}^{p} t_\ell \left(\sum_{i=1}^{m}\sum_{j=1}^{n} d^1_{ij\ell} x^*_i y^*_j + \sum_{i=1}^{m}\sum_{k=1}^{s} d^2_{ik\ell} x^*_i z^*_k + \sum_{j=1}^{n}\sum_{k=1}^{s} d^3_{jk\ell} y^*_j z^*_k\right),$$

Taking into account that $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = \sum_{k=1}^{s} z_k = \sum_{\ell=1}^{p} t_\ell = 1$, we have

$$F_1(x^*, y^*, z^*, t^*) \geq F_1(x, y^*, z^*, t^*),\ \forall x \in S_m,$$
$$F_2(x^*, y^*, z^*, t^*) \geq F_2(x^*, y, z^*, t^*),\ \forall y \in S_n,$$
$$F_3(x^*, y^*, z^*, t^*) \geq F_3(x^*, y^*, z, t^*),\ \forall z \in S_s,$$
$$F_4(x^*, y^*, z^*, t^*) \geq F_4(x^*, y^*, z^*, t),\ \forall t \in S_p.$$

which shows that (x^*, y^*, z^*, t^*) is a Nash equilibrium. The proof is complete. □

We consider the following optimization problem:

$$F(u) = F_1(x,y,z,t) + F_2(x,y,z,t) + F_3(x,y,z,t) + F_4(x,y,z,t) - \alpha - \beta - \gamma - \delta \uparrow \max_u \tag{8}$$

$$u = (x,y,z,t,\alpha,\beta,\gamma,\delta) \in S_m \times S_n \times S_s \times S_p \times \mathbb{R}^4, \tag{9}$$

$$(y,z,t,\alpha) \in X,\ (x,z,t,\beta) \in Y,\ (x,y,z,\gamma) \in Z,\ (x,y,z,\delta) \in T, \tag{10}$$

where the sets X, Y, Z and T obey the conditions:

$$X = \left\{ (y,z,t,\alpha) \in \mathbb{R}^{n+\ell+p+1} \left| \sum_{j=1}^{n}\sum_{k=1}^{s} a^1_{ijk} y_j z_k + \sum_{j=1}^{n}\sum_{\ell=1}^{p} a^2_{ij\ell} y_j t_\ell + \right. \right.$$
$$\left. + \sum_{k=1}^{s}\sum_{\ell=1}^{p} a^3_{ik\ell} z_k t_\ell \le \alpha e_m \right\},$$

$$Y = \left\{ (x,z,t,\beta) \in \mathbb{R}^{m+\ell+p+1} \left| \sum_{i=1}^{m}\sum_{k=1}^{s} b^1_{ijk} x_i z_k + \sum_{i=1}^{m}\sum_{\ell=1}^{p} b^2_{ij\ell} x_i t_\ell + \right. \right.$$
$$\left. + \sum_{k=1}^{s}\sum_{\ell=1}^{p} b^3_{jk\ell} z_k t_\ell \le \beta e_n \right\},$$

$$Z = \left\{ (x,y,t,\gamma) \in \mathbb{R}^{m+n+p+1} \left| \sum_{i=1}^{m}\sum_{j=1}^{n} c^1_{ijk} x_i y_j + \sum_{i=1}^{m}\sum_{\ell=1}^{p} c^2_{ik\ell} x_i t_\ell + \right. \right.$$
$$\left. + \sum_{j=1}^{n}\sum_{\ell=1}^{p} c^3_{jk\ell} y_j t_\ell \le \gamma e_s \right\}, \tag{11}$$

$$T = \left\{ (x,y,z,\delta) \in \mathbb{R}^{m+n+\ell+1} \left| \sum_{i=1}^{m}\sum_{j=1}^{n} d^1_{ij\ell} x_i y_j + \sum_{i=1}^{m}\sum_{k=1}^{s} d^2_{ik\ell} x_i z_k + \right. \right.$$
$$\left. + \sum_{j=1}^{n}\sum_{k=1}^{s} d^3_{jk\ell} y_j z_k \le \delta e_p \right\}$$

and $e_q = (1,1,\ldots,1)^T, \quad q = m,n,s,p.$

Theorem 2. *The strategy* (x^*, y^*, z^*, t^*) *is the Nash equilibrium in the game* $\Gamma(A,B,C,D)$ *if and only if there exist numbers* $(\alpha_*, \beta_*, \gamma_*, \delta_*)$ *such that* $u^* = (x^*, y^*, z^*, t^*, \alpha_*, \beta_*, \gamma_*, \delta_*) \in \mathbb{R}^{m+n+s+p+4}$ *is a global solution to problem (8)-(11).*

Proof. Necessity: Suppose that (x^*, y^*, z^*, t^*) is a Nash equilibrium. Choose scalars $\alpha^*, \beta^*, \gamma^*, \delta^*$ as: $\alpha^* = F_1(x^*, y^*, z^*, t^*)$, $\beta^* = F_2(x^*, y^*, z^*, t^*)$, $\gamma^* = F_3(x^*, y^*, z^*, t^*)$ and $\delta^* = F_4(x^*, y^*, z^*, t^*)$.
We show that the vector $(x^*, y^*, z^*, t^*, \alpha^*, \beta^*, \gamma^*, \delta^*)$ is a solution of the problem (8)-(11). First, we show that $(x^*, y^*, z^*, t^*, \alpha^*, \beta^*, \gamma^*, \delta^*)$ is a feasible point for the problem (8).
By the definition of a Nash equilibrium, we have

$$F_1(x^*, y^*, z^*, t^*) \geq F_1(x, y^*, z^*, t^*), \quad \forall x \in S_m,$$

$$F_2(x^*, y^*, z^*, t^*) \geq F_2(x^*, y, z^*, t^*), \quad \forall y \in S_n,$$

$$F_3(x^*, y^*, z^*, t^*) \geq F_3(x^*, y^*, z, t^*), \quad \forall z \in S_s,$$

$$F_4(x^*, y^*, z^*, t^*) \geq F_4(x^*, y^*, z^*, t), \quad \forall t \in S_p.$$

The rest of the constraints is satisfied since $x \in S_m,\ y \in S_n$ and $z \in S_s, t \in S_p$. It means that $(x^*, y^*, z^*, t^*, \alpha^*, \beta^*, \gamma^*, \delta^*)$ is a feasible point. Choose any $x \in D_m,\ y \in D_n,\ z \in D_s,\ t \in D_p$, and multiply inequalities in (11) by $x_i,\ y_j,\ z_k$ and t_ℓ respectively. Summing up these inequalities, we obtain

$$F_1(x, y, z, t) \leq F_1(x^*, y^*, z^*, t^*) = \alpha^*,$$

$$F_2(x, y, z, t) \leq F_2(x^*, y^*, z^*, t^*) = \beta^*,$$

$$F_3(x, y, z, t) \leq F_3(x^*, y^*, z^*, t^*) = \gamma^*,$$

$$F_4(x, y, z, t) \leq F_4(x^*, y^*, z^*, t^*) = \delta^*.$$

Hence, we get

$$F(x, y, z, t) \leq F(x^*, y^*, z^*, t^*) = F_1(x^*, y^*, z^*, t^*) + F_2(x^*, y^*, z^*, t^*) + \\ + F_3(x^*, y^*, z^*, t^*) + F_4(x^*, y^*, z^*, t^*) - \alpha^* - \beta^* - \gamma^* - \delta^* \leq 0.$$

for all $x \in D_m,\ y \in D_n,\ z \in D_s$ and $t \in D_p$.
But with $\alpha^* = F_1(x^*, y^*, z^*, t^*)$, $\beta^* = F_2(x^*, y^*, z^*, t^*)$, and $\gamma^* = F_3(x^*, y^*, z^*, t^*)$, $\delta^* = F_4(x^*, y^*, z^*, t^*)$ we have $F(x^*, y^*, z^*, t^*, \alpha^*, \beta^*, \gamma^*, \delta^*) = 0$ Hence, the point $(x^*, y^*, z^*, t^*, \alpha^*, \beta^*, \gamma^*, \delta^*)$ is a solution of the problem (8)-(11).

Sufficiency: Now we have to show reverse, namely, that any solution of the problem (8)-(11) must be a Nash equilibrium. Let $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ be any solution of the problem (8)-(11).
We show that $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ must be a Nash equilibrium of the game. Since $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ is a feasible point, we have

$$\sum_{j=1}^{n}\sum_{k=1}^{s} a^1_{ijk}\bar{y}_j\bar{z}_k + \sum_{j=1}^{n}\sum_{\ell=1}^{p} a^2_{ij\ell}\bar{y}_j\bar{t}_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} a^3_{ik\ell}\bar{z}_k\bar{t}_\ell \leq \bar{\alpha},\ i = 1, \dots, m, \tag{12}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{s} b^1_{ijk}\bar{x}_i\bar{z}_k + \sum_{i=1}^{m}\sum_{\ell=1}^{p} b^2_{ij\ell}\bar{x}_i\bar{t}_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} b^3_{jk\ell}\bar{z}_k\bar{t}_\ell \leq \bar{\beta},\ j = 1, \dots, n, \tag{13}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c^1_{ijk}\bar{x}_i\bar{y}_j + \sum_{i=1}^{m}\sum_{\ell=1}^{p} c^2_{ik\ell}\bar{x}_i\bar{t}_\ell + \sum_{j=1}^{n}\sum_{\ell=1}^{p} c^3_{jk\ell}\bar{y}_j\bar{t}_\ell \leq \bar{\gamma},\ k = 1, \dots, s, \tag{14}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} d^1_{ij\ell}\bar{x}_i\bar{y}_j + \sum_{i=1}^{m}\sum_{k=1}^{s} d^2_{ik\ell}\bar{x}_i\bar{z}_k + \sum_{j=1}^{n}\sum_{k=1}^{s} d^3_{jk\ell}\bar{y}_j\bar{z}_k \le \bar{\delta},\ \ell = 1,\dots,p. \qquad (15)$$

Now we multiply inequality (12) by x_i, (13) by y_j, (14) by z_k and (15) by t_ℓ respectively. Then we sum up these inequalities and obtain:

$$F_1(\bar{x},\bar{y},\bar{z},\bar{t}) \le \bar{\alpha},$$

$$F_2(\bar{x},\bar{y},\bar{z},\bar{t}) \le \bar{\beta},$$

$$F_3(\bar{x},\bar{y},\bar{z},\bar{t}) \le \bar{\gamma},$$

$$F_4(\bar{x},\bar{y},\bar{z},\bar{t}) \le \bar{\delta}.$$

Adding these inequalities, we obtain

$$F(\bar{x},\bar{y},\bar{z},\bar{t},\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}) = F_1(\bar{x},\bar{y},\bar{z},\bar{t}) + F_2(\bar{x},\bar{y},\bar{z},\bar{t}) + F_3(\bar{x},\bar{y},\bar{z},\bar{t}) +$$

$$+F_4(\bar{x},\bar{y},\bar{z},\bar{t}) - \bar{\alpha} - \bar{\beta} - \bar{\gamma} - \bar{\delta} \le 0. \qquad (16)$$

We know that at a Nash equilibrium $F(\bar{x},\bar{y},\bar{z},\bar{t},\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}) = 0$. Since $(\bar{x},\bar{y},\bar{z},\bar{t},\bar{\alpha},\bar{\beta},$ $\bar{\gamma},\bar{\delta})$ is also a solution, $F(\bar{x},\bar{y},\bar{z},\bar{t},\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta})$ be equal to zero:

$$F(\bar{x},\bar{y},\bar{z},\bar{t},\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}) = (F_1(\bar{x},\bar{y},\bar{z},\bar{t}) - \bar{\alpha}) + \big(F_2(\bar{x},\bar{y},\bar{z},\bar{t}) - \bar{\beta}\big) + (F_3(\bar{x},\bar{y},\bar{z},\bar{t}) - \bar{\gamma}) +$$

$$+ \big(F_4(\bar{x},\bar{y},\bar{z},\bar{t}) - \bar{\delta}\big) = 0.$$

Consequently,

$$\begin{cases} F_1(\bar{x},\bar{y},\bar{z},\bar{t}) = \bar{\alpha} \\ F_2(\bar{x},\bar{y},\bar{z},\bar{t}) = \bar{\beta} \\ F_3(\bar{x},\bar{y},\bar{z},\bar{t}) = \bar{\gamma} \\ F_4(\bar{x},\bar{y},\bar{z},\bar{t}) = \bar{\delta} \end{cases}$$

Since a point $(\bar{x},\bar{y},\bar{z},\bar{t},\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta})$ feasible, we can write the constraints (12)-(15) as follows:

$$\begin{cases} F_1(\bar{x},\bar{y},\bar{z},\bar{t}) \le F_1(x,\bar{y},\bar{z},\bar{t}), & \forall x \in S_m, \\ F_2(\bar{x},\bar{y},\bar{z},\bar{t}) \le F_2(\bar{x},y,\bar{z},\bar{t}), & \forall y \in S_n, \\ F_3(\bar{x},\bar{y},\bar{z},\bar{t}) \le F_3(\bar{x},\bar{y},z,\bar{t}), & \forall z \in S_s, \\ F_4(\bar{x},\bar{y},\bar{z},\bar{t}) \le F_4(\bar{x},\bar{y},\bar{z},t), & \forall t \in S_p. \end{cases}$$

Now taking into account the above results, by definition 1, we conclude that the point $(\bar{x},\bar{y},\bar{z},\bar{t})$ is a Nash equilibrium which completes the proof. □

Thus, finding Nash equilibrium reduces to solving problem (8)-(11). In order to solve problem (8)-(11), we apply a global search method proposed in (Gornov and Zarodnyuk, 2014).

3. The Curvilinear Multistart Algorithm (Enkhbat et al., 2016; Gornov and Zarodnyuk, 2014)

To solve problems (8)-(11), we use curvilinear multistart algorithm. The algorithm was originally developed for solving box-constrained optimization problems. To solve the original constrained problem (8)-(11), we transform it to a more simple box-contrained problem with penalties for original constraints. Denote:

$$G_1^i(y,z,t,\alpha) = \sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}^1 y_j z_k + \sum_{j=1}^{n}\sum_{\ell=1}^{p} a_{ij\ell}^2 y_j t_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} a_{ik\ell}^3 z_k t_\ell - \alpha \le 0, \quad i = 1,\dots,m, \tag{17}$$

$$G_2^j(x,z,t,\beta) = \sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk}^1 x_i z_k + \sum_{i=1}^{m}\sum_{\ell=1}^{p} b_{ij\ell}^2 x_i t_\ell + \sum_{k=1}^{s}\sum_{\ell=1}^{p} b_{jk\ell}^3 z_k t_\ell - \beta \le 0, \quad j = 1,\dots,n, \tag{18}$$

$$G_3^k(x,y,t,\gamma) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk}^1 x_i y_j + \sum_{i=1}^{m}\sum_{\ell=1}^{p} c_{ik\ell}^2 x_i t_\ell + \sum_{j=1}^{n}\sum_{\ell=1}^{p} c_{jk\ell}^3 y_j t_\ell - \gamma \le 0, \quad k = 1,\dots,s, \tag{19}$$

$$G_4^l(x,y,z,\delta) = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij\ell}^1 x_i y_j + \sum_{i=1}^{m}\sum_{k=1}^{s} d_{ik\ell}^2 x_i z_k + \sum_{j=1}^{n}\sum_{k=1}^{s} d_{jk\ell}^3 y_j z_k - \delta \le 0, \quad \ell = 1,\dots,p, \tag{20}$$

and

$$\begin{aligned} H_1(x) &= \sum_{i=1}^{m} x_i - 1 = 0,\ x_i \ge 0,\ i = 1,\dots,m,\\ H_2(y) &= \sum_{j=1}^{n} y_j - 1 = 0,\ y_j \ge 0,\ j = 1,\dots,n,\\ H_3(z) &= \sum_{k=1}^{s} z_k - 1 = 0,\ z_k \ge 0,\ k = 1,\dots,s,\\ H_4(t) &= \sum_{\ell=1}^{p} t_\ell - 1 = 0,\ t_\ell \ge 0,\ \ell = 1,\dots,p. \end{aligned} \tag{21}$$

$N = (m+n+s+p+4)$ and $u = (x,y,z,t,\alpha,\beta,\gamma,\delta) \in \mathbb{R}^N$, then our original problem can be solved as a series of box-contrained problems:

$$\min_{u\in D} f(u) = -F(u) + \mu_1^k \tilde{g}(u) + \mu_2^k \tilde{h}(u), \tag{22}$$

where

$$\tilde{g}(u) = \sum_{i=1}^{m} g_1^i + \sum_{j=1}^{n} g_2^i + \sum_{k=1}^{s} g_3^i + \sum_{l=1}^{p} g_4^i,$$

$$\tilde{h}(u) = \sum_{i=1}^{4} (H_i(u))^2,$$

$$g_i^j = \max(0, G_i^j(u))^2,$$

and the feasible set $D \subset \mathbb{R}^N$ is a simple box:

$$\begin{array}{ll} 0 \le x_i \le 1, i = \overline{1,m}; & 0 \le \alpha \le \overline{\alpha}; \\ 0 \le y_j \le 1, j = \overline{1,n}; & 0 \le \beta \le \overline{\beta}; \\ 0 \le z_k \le 1, k = \overline{1,s}; & 0 \le \gamma \le \overline{\gamma}; \\ 0 \le t_l \le 1, l = \overline{1,p}; & 0 \le \delta \le \overline{\delta}. \end{array}$$

The $\tilde{g}(u)$ and $\tilde{h}(u)$ are the penalty functions; μ_1 and μ_2 are the penalty coefficients. In each iteration of the optimization method, we increase the penalty coefficients, solve the problem and use the solution as the initial guess for the next iteration. Details are given in the **Algorithm Local**.

In order to desribe our global search algorithm, we also need to introduce the following definition:

Definition 2. A point $u^0 \in D$ is said to be a convex point with respect to a direction $p \in \mathbb{R}^N$, if the following condition holds:

$$\langle \frac{\partial^2 f(u^0)}{\partial u^2} p, p \rangle > 0 \tag{23}$$

where $\langle , \rangle$ denote the scalar product of two vectors in $\mathbb{R}^N$. Construct a line $u(t) = u^0 + hp, h \in \mathbb{R}_+$. It is clear that if the point u^0 is convex then there exist a positive h^*such that:

$$f(u^0 + hp) > f(u^0), \quad \forall h \in (0, h^*).$$

We can note that an approximate value of (23) can be computed for a sufficiently small h as follows:

$$\langle \frac{\partial^2 f(u^0)}{\partial u^2} p, p \rangle \approx \frac{1}{h^2} \left(f(u^0 + 2hp) - 2f(u^0 + hp) + f(u^0) \right). \tag{24}$$

To to solve problem (22), we propose a method of a global optimization which combines the ideas of the classical multistart and an estimation of the convexity degree of the starting point (Gornov and Zarodnyuk, 2014). The multistart idea remains to be one of the most popular among global optimization approaches. Despite the vociferous criticism of experts, it is sufficient to arrange two-level computational process for the successful solution of a simple nonconvex extremal problems. On the upper level we generate a random starting approximation, and on the lower level we perform the descent to a different extremum with the use of local algorithms.

We rely on the hypothesis that the algorithm starts from a local descent algorithm from a point in which the function is nonconvex is less computationally efficient, and it is preferable to use it as a starting point in which the function is convex. Since for multidimensional functions one can not determine whether the function is convex or not for sure, we resort to heuristic evaluation, which we call the convexity degree. The methodology for such numerical investigation of a convexity property was proposed in (Gornov and Zarodnyuk, 2014). The idea of the algorithm for testing the convexity of a point is to repeatedly generate random directions passing through this point and check the convexity with respect to chosen directions.

Estimation of a convexity degree of the function is calculated as the ratio of the number of "points of convexity" to the total number of investigated sample points.

Algorithm Global

Step 1. Set initial values for $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, $\overline{\delta}$, $k := 0$, $k^{\max} > 0$.
Step 2. Set $k := k + 1$.
Step 3. Generate a random point $u^k \in D$.
Step 4. Check the point u^k for the convexity by **Algorithm Convex** and if it is not convex, then go to **Step 3**.
Step 5. Start **Algorithm Local** from the point u^k to find a local solution $\tilde{u}^k$.
Step 6. If $f(\tilde{u}^k) < f(u^*)$, then $u^* := \tilde{u}^k$.
Step 7. If $k < k^{\max}$, then go to **Step 2**, otherwise u^* is an approximate global solution.

Algorithm Local

Step 1 Set penalty coefficients $\mu_1 := 1$ and $\mu_2 := 10$.
Step 2 Solve problem (22) from the starting point u^k with some local optimization method and place solution into $\tilde{u}^k$.
Step 3 Compare the α, β, γ, and δ values with their upper boundaries and increase them if necessary. For example, if $\alpha = \overline{\alpha}$, then $\overline{\alpha} := K \cdot \overline{\alpha}$, where $K > 1$ and go to **Step 2**.
Step 4 Compute the penalty value: $P := \tilde{g}(\tilde{u}^k) + \tilde{h}(\tilde{u}^k)$. If $P \geq \varepsilon_P$, then increase penalty coefficents $\mu_1 := 10\mu_1$, $\mu_2 := 10\mu_2$ and go to **Step 2**.
Step 5 Point $\tilde{u}^k$ is an approximate local solution.

Algorithm Convex

Step 1 Set $j := 0$, $j^{\max} > 0$, $h > 0$, $f^0 = f(u^k)$.
Step 2 Set $j := j + 1$ and generate a random direction $p^j \in \mathbb{R}^N$.
Step 3 Normalize the direction: $p^j := p^j / \|p^j\|_2$.
Step 4 Compute the values to check the point of convexity (see (24)): $f^1 = f(u^k + hp^j)$, $f^2 = f(u^k + 2hp^j)$,$C = (f^0 + f^2 - f^1)/h^2$.
Step 5 If $C < 0$, then u^k is not convex, exit.
Step 6 If $j < j^{\max}$, then go to **Step 2**.
Step 7 The point u^k is convex, exit.

4. Numerical Experiment

The proposed algorithm was tested on several four-players triple games. In all cases, Nash equilibrium points were found successfully. The following test problems were considered and solved by our algorithm on the computer with Intel Core i5-2400

CPU (3.1 GHz), 4 GB RAM.

The problems 2-4 were created by the well-known (GAMUT) generator. The generated game files and our solutions can be found in (DATA). For example, Problems 1 has a matrix payoffs a dimensions $2 \times 2 \times 2$.

Problem 1 ($2 \times 2 \times 2 \times 2$) Some points of the Nash equilibrium are:

F^*	x^*	y^*	z^*	t^*	α^*	β^*	γ^*	δ^*
0	(0, 1)	(1, 0)	(0, 1)	(1, 0)	1	2	3	1
0	(0, 1)	(1, 0)	(0, 1)	(0.95, 0.05)	0.8	1.7	1.7	1
0	(0, 1)	(1, 0)	(0, 1)	(0.9, 0.1)	0.6	1.4	2.8	1
0	(0, 1)	(1, 0)	(0, 1)	(0.85, 0.15)	0.4	1.1	2.7	1
0	(0, 1)	(1, 0)	(0, 1)	(0.8, 0.2)	0.2	0.8	2.6	1
0	(0, 1)	(1, 0)	(0, 1)	(0.75, 0.25)	0	0.5	2.5	1

Problem 2. GAMUT Random Game $3 \times 4 \times 5 \times 6$:

F^*	x^*, y^*, z^*, t^*	$\alpha^*, \beta^*, \gamma^*, \delta^*$
$1.43 \cdot 10^{-2}$	$x^* = (9.94, 0, 0.06)$	$\alpha^* = 48.57$
	$y^* = (0.34, 0.21, 0, 0.45)$	$\beta^* = 40.14$
	$z^* = (0, 0.26, 0.44, 0.3, 0)$	$\gamma^* = 42.30$
	$t^* = (0, 0, 0.18, 0.23, 0, 0.59)$	$\delta^* = 52.99$
$1.75 \cdot 10^{-2}$	$x^* = (0.06, 0.26, 0.68)$	$\alpha^* = 44.74$
	$y^* = (0.18, 0.5, 0.18, 0.14)$	$\beta^* = 50.26$
	$z^* = (0, 0.15, 0, 0.66, 0.19)$	$\gamma^* = 55.83$
	$t^* = (0.31, 0.09, 0.05, 0, 0.29, 0.26)$	$\delta^* = 53.32$
$1.68 \cdot 10^{-2}$	$x^* = (0.43, 0.57, 0)$	$\alpha^* = 50$
	$y^* = (0.07, 0, 0.23, 0.7)$	$\beta^* = 55.44$
	$z^* = (0.4, 0.04, 0, 0.20, 0.36)$	$\gamma^* = 44.04$
	$t^* = (0.53, 0.47, 0, 0, 0, 0)$	$\delta^* = 50.26$

Problem 3. GAMUT Random Game $6 \times 4 \times 3 \times 6$:

F^*	x^*, y^*, z^*, t^*	$\alpha^*, \beta^*, \gamma^*, \delta^*$
$8.69 \cdot 10^{-3}$	$x^* = (0.35, 0, 0.47, 0, 0, 0.18)$	$\alpha^* = 56$
	$y^* = (0.08, 0.37, 0, 0.55)$	$\beta^* = 50.23$
	$z^* = (0.25, 0.26, 0.49)$	$\gamma^* = 53.66$
	$t^* = (0, 0, 0, 0, 0.25, 0.75)$	$\delta^* = 49.08$
$1.74 \cdot 10^{-2}$	$x^* = (0, 0.22, 0.1, 0.37, 0.31, 0)$	$\alpha^* = 54.52$
	$y^* = (0.34, 0.25, 0.41, 0)$	$\beta^* = 45.94$
	$z^* = (0.47, 0.53, 0)$	$\gamma^* = 49.07$
	$t^* = (0, 1, 0, 0, 0, 0)$	$\delta^* = 59.75$
$1.99 \cdot 10^{-2}$	$x^* = (0, 0.52, 0, 0, 0.09, 0.39)$	$\alpha^* = 52.65$
	$y^* = (0, 0.91, 0.09, 0)$	$\beta^* = 56.1$
	$z^* = (0.2, 0.5, 0.3)$	$\gamma^* = 52.11$
	$t^* = (0.5, 0.5, 0, 0, 0, 0)$	$\delta^* = 57.37$

Problem 4. GAMUT Random Game $6 \times 8 \times 4 \times 5$:

F^*	x^*, y^*, z^*, t^*	$\alpha^*, \beta^*, \gamma^*, \delta^*$
$-6.97 \cdot 10^{-2}$	$x^* = (0, 0.11, 0.12, 0.02, 0.25, 0.5)$	$\alpha^* = 53.39$
	$y^* = (0, 0.18, 0, 0.13, 0.46, 0.1, 0.13, 0)$	$\beta^* = 53.20$
	$z^* = (0, 0, 0.55, 0.45)$	$\gamma^* = 51.81$
	$t^* = (0.5, 0.02, 0.22, 0.26, 0)$	$\delta^* = 56.87$
$1.09 \cdot 10^{-2}$	$x^* = (0, 0.16, 0.35, 0.45, 0.02, 0.02)$	$\alpha^* = 45.22$
	$y^* = (0, 0, 0.16, 0.22, 0, 0.49, 0.13, 0)$	$\beta^* = 57.64$
	$z^* = (0.53, 0.26, 0, 0.21)$	$\gamma^* = 47.8$
	$t^* = (0.2, 0.23, 0.19, 0.06, 0.32)$	$\delta^* = 54.14$
$1.45 \cdot 10^{-4}$	$x^* = (0, 0.31, 0.23, 0.46, 0, 0)$	$\alpha^* = 48.68$
	$y^* = (0.18, 0, 0, 0, 0, 0.08, 0.18, 0.56)$	$\beta^* = 55.32$
	$z^* = (0, 0.2, 0.8, 0)$	$\gamma^* = 45.87$
	$t^* = (0.17, 0, 0, 0.61, 0.22)$	$\delta^* = 47.96$

5. Conclusion

We examined the nonzero sum four person triple game from a viewpoint of the global optimization. Finding a Nash equilibrium of the game was reduced to a global optimization problem. To find the equilibrium points, we propose a method that combines the ideas of classical multistart and an estimation of a convexity of points. This method was examined numerically on some test problems generated by (GAMUT) and found solutions in all cases.

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GAMUT is a site of game generators. `http://gamut.stanford.edu`

DATA `https://www.dropbox.com/sh/sd84lbisy5vtifn/AAB2PNVWONtK56egStv8c-Vea?dl=0`