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The Problem of Supply Chain Profit Maximization Using Sales Rebate Contract

Irina Berezinets, Margarita Meshkova, Natalia Nikolchenko

St. Petersburg State University,
7/9 Universitetskaya nab., St. Petersburg, 199034, Russia
berezinets@gsom.pu.ru
rita.meshkova@gmail.com
st057455@student.spbu.ru

Abstract The paper considers the problem of supply chain profit maximization using the sales rebate contract. The problem solving is proposed for the two-echelon supply chain model with risk-neutral partners and the assumption of triangular distributed demand. It was shown that the sales rebate contract is not coordinating, as it does not provide the individual rationality for the supplier. The authors considered conditional coordination of the supply chain with sales-rebate contract, when the expected profits of the supply chain and the retailer reach their maximum, and the supplier's expected profit is greater than for the case of the wholesale price contract. It can be argued that the sales-rebate contract implementation under certain conditions is beneficial for both partners involved in the supply chain and provides the maximum of the supply chain expected profit. It was approved that the problem of supply chain profit maximization can be solved using the sales rebate contract.

Keywords: supply chain, profit, coordinating contract, sales rebate contract.

1. Introduction

The total financial result of a supply chain, such as profit, depends on the decisions made by each participant of the supply chain. At the same time, both partners involved in the supply chain seek to maximize their own profits, that often has a negative impact on each of partners, and, consequently, on the total value of supply chain's profits. However, it is possible to motivate supply chain partners to make decisions to maximize supply chain profit and to achieve an acceptable profit level for each partner through coordinating mechanisms, one of them are contracts. There are different types of contracts with the parameters determined as a result of supply chain partners' negotiations for instance, between a supplier and a retailer.

Finding contract parameters providing the maximum of supply chain profit could be a problem, especially given the fact that supply chain partners often do not have complete information. Even if we assume that the supply chain partners operate under condition of complete information, the demand is a random variable. Only after evaluation of the demand distribution, solving the problem of supply chain coordination under assumptions can be carried out.

The problem of determining optimal contract parameters providing the maximum of supply chain profit is quite relevant today. Often, the process of managing contractual relations between organizations occurs spontaneously, without any generally accepted rules and procedures. According to the latest study by Aberdeen

Research Group, clear regulation of contractual relations permits companies savings on transaction costs by 80%. Despite this, only 48% of organizations have a centralized contract management process. As for the research interest in this topic, there are many studies on contracts as coordination mechanisms, ranging from fundamental works such as the work of O. Williamson's "The Economic Institutions of Capitalism. Firms, Markets, Relational Contracting" and ending with studies considering the problem of supply chain coordination using certain types of contracts – T.A. Taylor, "Supply Chain Coordination Under Channel Rebates with Sales Effort Effects" (2002); G.P. Cachon, "Supply Chain Coordination with Contracts" (2003) and Chun-Hung Chiu et al. "Sales Rebate Contracts in Fashion Supply Chains" (2012). However, there are not many studies where the proposed algorithms for constructing a coordinating contract would be tested on real data. One of such works is the research Chiu et al. (2012).

The aim of the research is to solve the problem of the expected profit maximization of the supply chain using coordinating contract, especially sales rebate contract. The study considers a supply chain consisting of two partners: a supplier and a retailer. Authors proposed an algorithm for constructing coordinating contract under specified assumptions, which was carried out on the data of the pharmaceutical company. The practical contribution of this work lies in the application of the constructed mathematical models of supply chain coordination for solving actual management problems, that is, profit maximization by determining the contract parameters. The algorithm for building a coordinating contract can also be used by managers of suppliers and retailers to determine the fairness of the conditions of existing contracts.

The paper is organized as follows. The first part deals with the contract as coordination mechanism. The second part discusses the sales rebate contract, especially its application in practice, provides an overview of existing research, which solve the problem of supply chain coordination with sales rebate contract. It also presents the problem solving of supply chain coordination problem using sales rebate contract under the assumption that demand is a random variable with a triangular distribution. The third part examines the case of the pharmaceutical supply chain, which consists of a company engaged in the supply of specific products to medical institutions – a retailer and a large international manufacturer – supplier. In conclusion, the results of the study were summed up.

2. Contract as a supply chain coordination mechanism

The term contract can be considered from both economic and legal points of view. The economic interpretation of the contract and the contractual nature of the company's business objectives is reflected in the study of Williamson (1985). The author explains the ubiquity of contractual relations as follows: the solution of any economic problem can be organized in several ways implying the conclusion of a contract (Williamson, 1985). In this case, the contract can be determined both by words (formal contract, explicit contract) and concluded in writing or orally, and by the actions of partners (implied contract, implicit contract). Williamson defines a contract as "an agreement between a buyer and a supplier in which the terms of an exchange are determined by three factors: price, asset specificity, and guarantees" (cited in Williamson, 1985). The conclusion of such an agreement usually begins with a negotiation process, the purpose of which is to determine the contract parameters (terms of exchange) acceptable to each party. The concept of acceptability implies that the expected profit of the company from participation in the contract with specific parameters will not be less than a certain exogenously established level (reservation profit). This level usually reflects the expected profit that a company can make by using alternative opportunities to place its resources, that is, the expected profit without participation in the contract in question (Bernstein and Marx, 2006). If the parties agree on such conditions, the contract is concluded; if the parties fail to agree, the contract is not concluded.

The contract as a physical object should also be considered from a legal point of view. Tambovtsev provides the following definition of this concept in civil law: contract is the "agreement of two or more persons concerning the establishment, amendment (redistribution) or the termination of certain rights, primarily the right of ownership on the property or otherwise" (cited on Tambovtsev, 2004, p. 24).

Contracts are one of the mechanisms for supply chain coordination. Depending on what is meant by coordination, there are different definitions of a coordinating contract. In this paper, we assume that under the assumption of risk neutrality of the chain's partners, the contract is a coordinating one if it motivates each partner to make such decisions providing the maximum of the expected profit of the supply chain. In other words, a contract is a coordinating if the contract parameters, determined by each partner on the basis of the condition of maximizing their winnings (expected profits), allow to obtain the maximum value of the expected profit of the supply chain. It should also be noted that coordination is achieved in case when the expected profit of the supply chain is maximum, the solution of the problem of supply chain expected profit maximization can be reduced to the problem of coordination.

The construction of any contract describing the interaction of supply chain partners takes place within a certain model, that is, some theoretical game. For supply chains, the terms of the game can be interpreted as follows:

- Players A and B – supply chain partners (e.g. supplier and retailer);
- The set of admissible strategies is the set of feasible solutions associated with the determination of the parameters of the contract (order quantity/production, sales price, wholesale price, rebate values);
- Payoff function – the expected profit of each partner per transaction under the assumption of partisipants risk neutrality.

Since this paper discusses the supply chain, consisting of a supplier and a retailer that sells products on the market, the sequence of events occurring in the game will be as follows (Cachon, 2003):

- The supplier offers the retailer the terms of the contract for the purchase of one type of product;
- The retailer accepts or rejects the proposed contract (the situation when the retailer does not accept the terms of the contract is not considered in this study);
- If the retailer accepts the terms of the contract, it chooses the volume of purchased products from the supplier (q);
- The supplier delivers q units to the retailer before the start of the sales season;
- The retailer sells products on the market in accordance with the implementation of demand within one sales season;

· The parties fulfill their obligations under the contract (money transfer takes place).

One implementation of the described sequence of events corresponds to a single transaction. This approach, first described in the work of Commons, involves the analysis of trade at the micro level of analysis (Commons, 1934) and the possibility of introducing the condition of profit additivity, that is, the profit of the supply chain consists of the profits of all partners included in the chain, in this case, the supplier and the retailer (Gan, 2011). Note that since the players in this game act consistently (first, the supplier offers the terms of the contract, and then the retailer responds to these conditions), the scenario of interaction between the retailer and the supplier corresponds to the Stackelberg model. The assumption that the first move in the game is made by the supplier does not affect the subsequent analysis of the coordinating contract, since the model does not consider the negotiation process, the features of which can affect the terms of the exchange (Cachon, 2003).

The Stackelberg model is used when players have different market power and as a result the decision-making process is consistent (Kogan, Tapiero, 2007). The player who first chooses his strategy is the leader. Then the second player – the follower – chooses his best response to the leader's move. Thus, the leader has an advantage, since he can optimize his target function taking into account the answer of the follower known to him in advance. It is worth noting that this situation is possible only if the leader knows the function of winning the follower (Kogan, Tapiero, 2007).

From a formal point of view, the Stackelberg model can be represented as follows. Consider a game that is played only by two players – A and B. Let

· Y_A – the set of admissible strategies of the player A;
· Y_B – the set of admissible strategies of the player B;
· $K_A(y_A,\ y_B)$ – the payoff function of player A;
· $K_B(y_A,\ y_B)$ – the payoff function of player B.

Suppose that the game is implemented under condition of complete information for both players. Denote by

· $y_B^* = y_B^R(y_A)$ – the best response of player B to player A's move, which is determined from the following condition: $\max\limits_{y_B} K_B(y_A,\ y_B) = K_B(y_A,\ y_B^*) = K_B\left(y_A,\ y_B^R(y_A)\right)$;
· $y_A^* = y_A^R(y_B^*) = y_A^R\left(y_B^R(y_A)\right)$ – the best strategy of player A, provided that the best response of player B is known in advance, which is determined from the following condition: $\max\limits_{y_A} K_A(y_A,\ y_B^*) = K_A(y_A^*,\ y_B^*) = K_A\left(y_A^*,\ y_B^R(y_A)\right)$.

A couple of strategies $(y_A^*,\ y_B^*)$ is a Stackelberg equilibrium. Note that the Stackelberg equilibrium is essentially a Nash equilibrium in the Stackelberg model.

Thus, the following conclusions can be drawn with regard to the interaction between the supplier and the retailer in the supply chain. If the solution of the game is a Nash equilibrium, it provides the maximum possible expected profit for each supply chain partner, since the implementation of such a solution is based on the principle of maximizing each partner's payoff function. If the solution is Pareto-optimal, it provides the maximum of supply chain expected profit. The validity of the reverse statement is ensured by the fulfillment of the profit additivity condition in case of partners are risk neutrality (Gan, 2011). The coordinating contract can be defined as a contract that motivates the supply chain partners to make their decisions that are the Nash equilibrium at the same time, and possess the property of Pareto optimality. In other words, the coordinating contract must fulfil the properties of individual and collective rationality.

Note that it is possible that there is no solution in the game that would be a Nash equilibrium. For the Stackelberg model, this situation can be interpreted as follows: it is impossible to determine a strategy for player A that would allow to maximize its payoff function with the known best response of player B. In this case, the condition of individual rationality for player A is not fulfilled, and therefore the contract under study is not coordinating. In this case, player A can get more expected profit than some set level under certain conditions. In such a situation, the contract is beneficial for each partner, so we consider the conditionally coordinated supply chain. If the contract is coordinating by definition, then there is unconditional coordination of the supply chain.

In order to determine the parameters of the coordinating contract, describing the interaction of two supply chain partners, it is necessary to perform the following steps:

1. Determine the parameters affecting the expected profit of the supply chain, in which the value of expected profit is maximum;
2. Determine the parameters affecting the individual functions of the expected profit of each partner, in which these expected profits are maximum, taking into account the parameters selected in the first step;
3. Make sure that the terms of the contract are acceptable for each partner, each partner receives the maximum of expected profit.

If the latter condition is not met for at least one partner, the contract is considered non-coordinating. However, in a situation where the parameters found and/or the corresponding restrictions on these parameters, the expected profit of each partner exceeds a certain set level, the contract conditionally coordinates the supply chain. In such a situation, the conclusion of the contract is beneficial for each partner in the supply chain. If none of the above situations occur when the parameters are found, then the contract with such parameters will probably not be concluded.

3. Sales rebate contract review

There are many different types of contracts that are used by companies, ranging from the simplest wholesale price contract to more complex contracts that combine several simple contract models. Researchers most often analyze the following types of contracts: wholesale-price contract; revenue-sharing contract; sales rebate contract; buy-back contract; quantity-flexibility contract; two-part tariff contract; quantity discount contract.

In practice, the choice of a particular type of contract depends on the ratio of possible benefits for partners and transaction costs. For example, the above types of contracts differ in the cost of monitoring the performance of these contracts. More complex contract models involve the information sharing (for example, the number of products sold), which affects the amount of payments of partners under the contract terms. The cost of verifying the accuracy of such information will affect the ability to enter into a particular type of contract. In addition, each type of contract motivates partners to perform different actions, for example, to buy/produce a larger volume of products, sell a larger volume of products, etc. Thus, the choice of the type of contract depends on the characteristics of the products and the characteristics of the specific industry in which the partners operate.

In this paper, we will consider the sales rebate contract, which was first described by Taylor. The main idea of this contract is as follows: the supplier pays the retailer a rebate per unit sold by the threshold of sales volume set by the supplier. There are two main types of sales rebate contract (Taylor, 2002): linear rebate involves the payment of a rebate per unit sold by the retailer; target rebate means the payment of a rebate per unit sold by the retailer in excess of the target sales volume set by the supplier.

The sales rebate contracts are common in various industries (personal computer manufacturing, automotive, fashion, pharmaceuticals), so they are of particular interest to researchers as supply chain coordination mechanisms. This contract was first described in (Taylor, 2002). The author explores the possibility of coordinating the supply chain, consisting of the manufacturer and the retailer, through a sales rebate contract in the framework of the model, when the retailer does not make additional efforts to increase sales. The main conclusion made by the author is that for this model, the use of a sales rebate contract helps to achieve supply chain coordination and a mutually beneficial outcome for both partners; while a linear sales rebate contract is non-coordinated (Taylor, 2002). Coordination in the study refers to the situation when the profit of the supply chain is maximum, and the win-win situation that is the situation when both partners receive more profit than when using the wholesale price contract with any available distribution of the supply chain profit between the partners (Taylor, 2002). Taylor (2002) also argues that if the retailer makes additional efforts to increase sales, the sales rebate contract will help to achieve supply chain coordination and a win-win situation for both partners only in conjunction with the terms of the contract with the possibility of repurchase (when unsold products are returned to the supplier for a fee) (Taylor, 2002). The same model is considered in the article (Krishnan et al., 2004). The authors consider the limitations of the contract with the possibility of repurchase and ways to overcome these limitations by complicating the contract models. One of these complications is the rebates for the retailer per unit sold in excess of a certain volume (markdown allowances). The combination of a contract with a repurchase option and rebates for the retailer allows achieving coordination (Krishnan et al., 2004).

Cachon considers various types of contracts, including sales rebate contracts, within the framework of two models. In the first model, the retail sales price (p) is fixed and is set by the supplier, in the second model the retail sales price (p) is the parameter that the retailer chooses. The author concludes that in the first model the sales rebate contract is coordinating under certain conditions; in the second model it is impossible to achieve coordination using this contract (Cachon, 2003; Cachon and Lariviere, 2005). Cachon believes that a contract is a coordinating one if the strategies implemented by the supply chain partners optimal for the supply chain are also Nash equilibrium. At the same time, such optimal strategies should be the unique Nash equilibrium, otherwise supply chain partners may decide that the chain is coordinated when implementing strategies that are not optimal for it (Cachon, 2003). In addition, Cachon considers the coordination chain using contracts in the framework of more complex models (model one supplier and multiple competing retailers, the model with the possibility of re-ordering in one selling season, etc.).

Studies of (Taylor, 2002) and (Cachon, 2003) are most important among research analyzed sales rebate contracts. Other authors in their papers often refer to them in the description of the contract or in determining the coordination of the supply chain. In general, the subsequent studies related to the considered type of contract can be divided into three groups depending on the analyzed aspect:

- Studies in which the sales rebate contract is considered in the framework of sophisticated models of supply chain (Chiu et al., 2011-b; Xing and Liu, 2012; Lan et al., 2015; Pu et al., 2017);
- Research that analyze different types of sales rebate contracts (Aydin and Porteus, 2009; Arcelus et al., 2012; Liao, 2013; Saha 2013);
- Studies addressed the supply chain coordination using the sales rebate contract and other coordination mechanisms (Wong and Leung, 2009; Chiu et al., 2011-a).

Among the research of the last year there are the studies where the authors propose a new modification of the sales rebate contract, allowing to achieve the supply chain coordination in the framework of more complex models (Heydari and Asl-Najafi, 2017; Genc and Giovanni, 2018; Muzaffar et al., 2018; Sainathan and Groenevelt, 2019). For example, Heydari, Asl-Najafi offer a "new" sales rebate contract that coordinates the supply chain when the retailer makes additional efforts to increase sales (Heydari and Asl-Najafi, 2017). One of the latest papers (Sainathan and Groenevelt, 2019) analyzes the interaction between the supplier and the retailer within the VMI coordination mechanism, as well as using various contracts. The authors also propose new modifications of existing contracts with the possibility of repurchase and contracts flexible in volume (Sainathan and Groenevelt, 2019).

4. Supply chain coordination with sales rebate contract under the assumption that demand has a triangular distribution

Let us solve supply chain coordination problem with the sales rebate contract. For this purpose, it is necessary to find parameters of a coordinating contract, that is, such a contract that ensures the fulfillment of both individual and collective rationality properties. In other words, the expected supply chain profit with these parameters should be maximum, and the expected profit of each supply chain partner is maximum possible under the assumption that each supply chain participant acts rationally and maximizes their expected profit.

Consider a supply chain consisting of two risk-neutral players: a supplier and a retailer that sell a single product. The supplier sells the product to the retailer that then sells it to the market within a single season. Suppose that the parties interact with each other under the target rebate contract, and also that the supplier has exceptional market power. It means that he offers the retailer a take-it-or-leave-it contract, that is, the conditions offered by the supplier are not discussed. The retailer can either accept such a contract or refuse to cooperate.

As the first step the supplier offers the retailer the following contract parameters: the wholesale price per unit (ω) and the amount of rebate (r) paid per unit sold above the established threshold (t). In response to the conditions offered by the supplier, the retailer chooses what volume of products (q) she should order to maximize her profit. The sales rebate contract implies that $t \leq q$. After signing the contract and the delivery of products, the retailer sells them in the market at the certain price per unit (p). In this model, the retail price per unit (p) is not discussed at the time of signing the contract and is fixed. If the retailer is unable to sell the entire volume of purchased products at the initial retail price (p), then she can sell the remaining products at the salvage value per unit (v).

Let us analyze the interaction between the retailer and the supplier within the framework of the Stackelberg model. Since the supplier (S) is the first to choose his strategy, he is a leader, and the retailer (R) is a follower. The supplier's strategy is to choose three parameters: $\omega,\ r,\ t$ – from the available set; the retailer chooses only the volume of purchased products – q – also from the available set (for example, in some contracts the minimum purchase volume is required). The expected profit for one transaction is considered as a payoff function both for the supplier and the retailer. Thus, to find the optimal solution for such a model, firstly it is necessary to find the optimal solution for the retailer (find the optimal purchase volume q_R^*), and then using the expression for q_R^*, optimize the supplier's expected profit function. The found solution is the Shtakelberg equilibrium that fulfills the condition of individual rationality.

The fulfillment of collective rationality property is ensured through the Pareto-optimality of the found solution. In this model the solution is Pareto optimal if the maximum of the expected supply chain profit is achieved (Gan, 2011). The parameter that affects the supply chain expected profit is the volume of purchased products (q). This parameter is chosen by the retailer. The parameters that influence the distribution of the expected supply chain profits between partners – $\omega,\ r,\ t$ – are determined by the supplier. Since the supplier has exceptional market power, in order to motivate the retailer to choose such a volume of purchased products (q_{SC}^*), at which the expected supply chain profit function ($E[\Pi_{SC}]$) reaches its maximum, he must choose such an amount of the wholesale prices (ω), so that at the optimal solution for the supply chain the retailer's expected profit function ($E[\pi_R]$) also reaches its maximum ($q_{SC}^* = q_R^*$). At the same time, with the help of two other parameters responsible for the distribution of the supply chain profit between partners – r and t – the supplier optimizes his expected profit function ($E[\pi_S]$). Thus, the construction of a coordinating contract (determination of the parameters of such a contract) for the considered interaction between the supplier and the retailer consists of the following steps:

1. Determination of the optimal purchase volume for the retailer (q_R^*);
2. Determination of the optimal purchase volume for the supply chain (q_{SC}^*);
3. Determination of the wholesale price value ω^*, at which the optimal solution for the retailer coincides with the optimal solution for the supply chain ($q_{SC}^*{=}q_R^*$);
4. Determination of the parameters r and t, at which the expected profit of the supplier ($E[\pi_S]$) is maximum for obtained q^* and ω^*.

To solve the supply chain coordination task with the sales rebate contract, the notations presented in the table 1 are used.

Table 1. Notations used for supply chain coordination problem solving

ω	Wholesale price per unit (c.u.)
r	Rebate value, which is paid by the supplier to the retailer per unit sold above the established threshold (c.u.)
t	Sales volume, in excess of which, the supplier pays the retailer a rebate per each unit sold above this volume (set by the supplier, pcs.)
q	Volume of products purchased by the retailer from the supplier (pcs.)
p	Retail price per unit (c.u.)
v	Salvage value per unit (c.u.)
c_S	Supplier's production costs per unit (c.u.)
c_R	Retailer's marginal costs per unit (c.u)
c	Supply chain total costs, $c = c_S + c_R$ (c.u.)
π_S	Supplier's profit for one transaction (c.u.)
π_R	Retailer's profit for one transaction (c.u.)
Π_{SC}	Supply chain profit for one transaction, $\Pi_{SC} = \pi_S + \pi_R$ (c.u)

Let us also denote additional prerequisites of the model and some restrictions on the parameters:

1. Both companies possess complete information to determine the optimal actions, including the costs (c_S, c_R), the retail price (p) and the salvage value (v);
2. Optimal actions are feasible for each company;
3. Both companies are risk-neutral, that is, they are aimed at maximizing the expected profits without taking risk into account;
4. The retailer has no possibility of additional replenishment of stocks within the one sales season;
5. The model does not consider the supplier's choice of the optimal production volume;
6. Reputational losses are not considered (reputational costs are zero, both for the supplier and for the retailer);
7. Demand distribution function is differentiable, strictly increasing;
8. The retail price is higher than the wholesale price; the wholesale price is higher than the supplier's production costs per unit:
$$0 < c_S < \omega < p.$$
9. The salvage value is lower than the supplier's production costs per unit:
$$v < c_S.$$

The solution of the supply chain coordination problem with the sales rebate contract in the framework of the described model begins with the assumption that the demand for the considered type of product is a random variable. Let ξ is the demand for some type of product, τ is the sales volume of this type of product. Suppose that $\tau = g(\xi)$, where

$$\tau = g(\xi) = \begin{cases} \xi, & 0 \le \xi < q \\ q, & \xi \ge q \end{cases}.$$

Let ξ be a continuous random variable, $f_\xi(x)$ is its probability density function, $F_\xi(x)$ is its distribution function. Then the expected sales volume, that is, the expected value of a random variable τ, can be calculated as follows:

$$E\left[\tau\right] = E\left[g\left(\xi\right)\right] = qF_\xi\left(q\right) - \int_0^q F_\xi\left(x\right)dx + q\left(1 - F_\xi\left(q\right)\right) = q - \int_0^q F_\xi\left(x\right)dx. \quad (1)$$

Based on the terms of the sales rebate contract, the profits of the supplier and the retailer depend on whether the latter can sell the volume exceeding the threshold t specified in the contract. Two cases follow from this statement: the actual sales volume is less than the level established by the contract ($0 \le \tau \le t$) and the actual sales volume is higher than the level established by the contract ($t < \tau \le q$).

The first case ($0 \le \tau \le t$)

The profits of the retailer, the supplier and the supply chain with $0 \le \tau \le t$ are:

$$\pi_R = p\tau + \left(q - \tau\right)v - c_R q - \omega q = \left(p - v\right)\tau - \left(\omega + c_R - v\right)q, \quad (2)$$

$$\pi_S = \omega q - c_S q = \left(\omega - c_S\right)q, \quad (3)$$

$$\mathit{\Pi}_{SC} = \pi_R + \pi_S = \left(p - v\right)\tau - \left(c - v\right)q. \quad (4)$$

It is worth noting, that in this case the supplier does not pay any rebate to the retailer, and the regulation of the supply chain partners' actions takes place within the framework of the wholesale price contract.

The expected profits of the retailer, the supplier and the supply chain, based on (2)–(4), are given by the following expressions:

$$E\left[\pi_R\right] = \left(p - v\right)\left(q - \int_0^q F_\xi\left(x\right)dx\right) - \left(\omega + c_R - v\right)q, \quad (5)$$

$$E\left[\pi_S\right] = \left(\omega - c_S\right)q, \quad (6)$$

$$E[\Pi_{SC}] = \left(p - v\right)\left(q - \int_0^q F_\xi\left(x\right)dx\right) - \left(c - v\right)q. \quad (7)$$

The second case ($t < \tau \le q$)

In the second case, when the actual sales volume exceeds the level established in the contract ($\boldsymbol{t < \tau \le q}$), the retailer receives the rebate from the supplier for each unit sold above this level. Then the expressions for the retailer's, the supplier's and supply chain profits are as follows:

$$\pi_R = p\tau + \left(q - \tau\right)v + r\left(\tau - t\right) - c_R q - \omega q = \\ = \left(p - v + r\right)\tau - \left(\omega + c_R - v\right)q - tr, \quad (8)$$

$$\pi_S = \omega q - c_S q - r\left(\tau - t\right) = -r\tau + \left(\omega - c_S\right)q + tr, \quad (9)$$

$$\Pi_{SC} = \pi_R + \pi_S = \left(p - v\right)\tau - \left(c - v\right)q. \quad (10)$$

The expected profits of the retailer, the supplier and the supply chain, based on (8)–(10), are given by the following expressions:

$$E\left[\pi_R\right] = \left(p - v + r\right)\left(q - \int_0^q F_\xi\left(x\right)dx\right) - \left(\omega + c_R - v\right)q - tr, \quad (11)$$

$$E\left[\pi_S\right]=\left(\omega-c_S\right)q-r\left(q-\int_0^q F_\xi\left(x\right)dx\right)+tr, \tag{12}$$

$$E[\Pi_{SC}]=\left(p-v\right)\left(q-\int_0^q F_\xi\left(x\right)dx\right)-\left(c-v\right)q. \tag{13}$$

Consider the problem of constructing a coordinating contract under the assumption that demand (the random variable ξ) has the triangular distribution with the range $[0,\ h]$.

The probability density function of the random variable ξ is given by the following expression:

$$f_\xi\left(x\right)=\begin{cases}0, & x<0,\\ \frac{2(h-x)}{h^2}, & 0\le x\le h,\\ 0, & x>h.\end{cases}$$

The distribution function of the random variable ξ has the following form:

$$F_\xi\left(x\right)=\begin{cases}0, & x<0,\\ 1-\frac{(h-x)^2}{h^2}, & 0\le x\le h,\\ 1, & x>h.\end{cases} \tag{14}$$

Consider the case when $h>q$. Let us find the expected value of the random variable τ ($E\left[\tau\right]$) for the considered triangular distribution, substituting the distribution function from the expression (14) into the expression (1):

$$E\left[\tau\right]=E\left[g\left(\xi\right)\right]=q-\int_0^q\left(1-\frac{\left(h-x\right)^2}{h^2}\right)dx=q-\left(q+\frac{\left(h-x\right)^3}{3h^2}\Big|_0^q\right)= \\ =\frac{h}{3}-\frac{\left(h-q\right)^3}{3h^2}. \tag{15}$$

Let us write expressions for the functions of the expected profits of the retailer, the supplier and the supply chain, if the actual sales volume is less than the level established by the contract ($0\le\tau\le t$):

$$E\left[\pi_R\right]=\left(p-v\right)\left(\frac{h}{3}-\frac{\left(h-q\right)^3}{3h^2}\right)-\left(\omega+c_R-v\right)q, \tag{16}$$

$$E\left[\pi_S\right]=\left(\omega-c_S\right)q, \tag{17}$$

$$E[\mathit{\Pi}_{SC}]=\left(p-v\right)\left(\frac{h}{3}-\frac{\left(h-q\right)^3}{3h^2}\right)-\left(c-v\right)q. \tag{18}$$

If the actual sales volume exceeds the level established by the contract ($t<\tau\le q$), the expressions for the expected profits of the retailer, the supplier and supply chain are:

$$E\left[\pi_R\right]=\left(p-v+r\right)\left(\frac{h}{3}-\frac{\left(h-q\right)^3}{3h^2}\right)-\left(\omega+c_R-v\right)q-tr, \tag{19}$$

$$E\left[\pi_S\right]=\left(\omega-c_S\right)q-r\left(\frac{h}{3}-\frac{\left(h-q\right)^3}{3h^2}\right)+tr, \tag{20}$$

$$\mathrm{E}[\Pi_{SC}] = (p-v)\left(\frac{h}{3} - \frac{(h-q)^3}{3h^2}\right) - (c-v)\,q. \tag{21}$$

In the case when the actual sales volume is less than the level established by the contract ($0 \le \tau \le t$), the actions of the supply chain parnters are regulated within the framework of the wholesale price contract, which is not coordinating. Let us turn to solving the problem of constructing a coordinating contract for the second case ($t < \tau \le q$). To do this, consider the 4 steps of constructing a coordinating sales rebate contract.

Determination of the optimal purchase volume for the retailer (q_R^*)

The retailer chooses the optimal purchase volume (q_R^*) after the supplier offers the following contract terms: the wholesale price (ω), the amount of rebate (r) and sales volume, in excess of which, the supplier pays the retailer a rebate per each unit sold above this volume (t). Let us find the maximum of the expected retailer's profit function $E[\pi_R]$ (q_R^*) for the considered case of the triangular distribution of the random variable ξ.

The first derivative of the expected sales volume $E[\tau]$ for q has the following form:

$$(E[\tau])' = 1 - F_\xi(q). \tag{22}$$

Find the first order derivative of the function $E[\pi_R]$ for q, using the expression (22):

$$\frac{dE[\pi_R]}{dq} = (p - v + r)(1 - F_\xi(q)) - (\omega + c_R - v).$$

The necessary extremum condition allows finding stationary points, which can then be explored to the maximum:

$$\frac{dE[\pi_R]}{dq} = (p - v + r)(1 - F_\xi(q)) - (\omega + c_R - v) = 0.$$

The stationary point of the function $E[\pi_R]$ (q_R^0) satisfies the following condition:

$$F_\xi\left(q_R^0\right) = \frac{p + r - \omega - c_R}{p - v + r}. \tag{23}$$

To test the sufficient condition of the extremum, with the help of which it is determined whether the found stationary point is a maximum, minimum or saddle point, we find the second order derivative of the function $E[\pi_R]$:

$$\frac{d^2E[\pi_R]}{dq^2} = (p - v + r)(-f_\xi(q)).$$

By the condition of the problem $p > v$, the distribution density function $f_\xi(x)$ takes only non-negative values. It follows that the second order derivative always takes only nonpositive values, in particular

$$\frac{d^2E[\pi_R]}{dq^2} = (p - v + r)\left(-f_\xi\left(q_R^0\right)\right) \le 0.$$

It can be concluded that the stationary point q_R^0 is the maximum point of the expected retailer's profit function $E[\pi_R]$ (q_R^*).

Substitute the distribution function (14) in the expression (23)

$$1-\frac{(h-q_R^*)^2}{h^2}=\frac{p+r-\omega-c_R}{p-v+r}.$$

Hence

$$(h-q_R^*)^2=h^2\left(1-\frac{p+r-\omega-c_R}{p-v+r}\right),$$

$$q_{R,1}^*=h\left(1-\sqrt{1-\frac{p+r-\omega-c_R}{p-v+r}}\right), \quad (24)$$

$$q_{R,2}^*=h\left(1+\sqrt{1-\frac{p+r-\omega-c_R}{p-v+r}}\right).$$

Note that $q_{R,2}^* > h$, and therefore this solution does not satisfy the introduced constraint $h > q$. Thus, the optimal purchase volume for the retailer is given only by the expression (24), $q_{R,1}^* = q_R^*$.

Determination of the optimal purchase volume for the supply chain (q_{SC}^*)

The optimal purchase volume for the supply chain (q_{SC}^*) is such an amount at which the expected profit of the supply chain reaches its maximum. The procedure for finding the maximum of the expected supply chain profit $E[\Pi_{SC}]$ (q_{SC}^*) is identical to the procedure of finding the maximum of the expected retailer's profit $E[\pi_R]$ (q_R^*) discussed above.

The first order derivative of the function $E[\Pi_{SC}]$ for q using the expression (22):

$$\frac{dE[\Pi_{SC}]}{dq}=(p-v)\left(1-F_\xi\left(q\right)\right)-(c-v).$$

The necessary extremum condition:

$$\frac{dE[\Pi_{SC}]}{dq}=(p-v)\left(1-F_\xi\left(q\right)\right)-(c-v)=0.$$

The condition for the stationary point of the function $E[\Pi_{SC}]$ (q_{SC}^0):

$$F_\xi\left(q_{SC}^0\right)=\frac{p-c}{p-v}. \quad (25)$$

The second order derivative of the function $E[\Pi_{SC}]$for q:

$$\frac{d^2E[\Pi_{SC}]}{dq^2}=(p-v)\left(-f_\xi\left(q_R^0\right)\right)\leq 0.$$

Thus, the stationary point (q_{SC}^0) is the maximum point of the expected supply chain profit function $E[\Pi_{SC}]$ (q_{SC}^*).

Find the maximum point of the expected supply chain profit function $E[\Pi_{SC}]$ (q_{SC}^*) for the considered case of the triangular distribution of the random variable ξ. To do this, we substitute the distribution function (14) into expression (25).

$$1-\frac{(h-q_{SC}^*)^2}{h^2}=\frac{p-c}{p-v}.$$

Hence

$$\left(h - q^*_{SC}\right)^2 = h^2 \left(1 - \frac{p - c}{p - v}\right),$$

$$q^*_{SC,1} = h \left(1 - \sqrt{1 - \frac{p - c}{p - v}}\right), \tag{26}$$

$$q^*_{SC,2} = h \left(1 + \sqrt{1 - \frac{p - c}{p - v}}\right).$$

Note that $q^*_{SC,2} > h$, and therefore, this solution does not satisfy the introduced constraint $h > q$. Thus, the optimal purchase volume for the supply chain will be specified only by the expression (26), $q^*_{SC,1} = q^*_{SC}$.

Determination of the wholesale price optimal value ω^*

The wholesale price optimal value ω^* is the value at which the optimal solution for the retailer coincides with the optimal solution for the supply chain ($q^*_{SC} = q^*_R$). In this case, the maximum of the expected profit of the retailer $E[\pi_R]$, and the expected profit of the supply chain $E[\Pi_{SC}]$ is reached at the same time.

Let us find the expression for ω^* for the considered case of the triangular distribution of the random variable ξ. To do this, we equate the expressions (24) and (26).

$$h \left(1 - \sqrt{1 - \frac{p + r - \omega - c_R}{p - v + r}}\right) = h \left(1 - \sqrt{1 - \frac{p - c}{p - v}}\right).$$

Hence:

$$(p - v)(p + r - \omega^* - c_R) = (p - c)(p - v + r),$$

$$\omega^* = \frac{-c_S p + c_S v - c_S r - c_R r + v r}{v - p} = c_S + \frac{c - v}{p - v} r. \tag{27}$$

Determination of the parameters r *and* t

In the previous steps of constructing a coordinating contract we determine the parameters q^* and ω^*, at which the condition of individual rationality for the retailer (the retailer's expected profit is maximum) and the condition of collective rationality for the supply chain (the expected supply chain's profit is maximum) are met. Now it is necessary to achieve the fulfillment of the individual rationality property for the supplier, that is, to find such parameters r and t, at which the expected profit of the supplier $E[\pi_S(r, t)]$ is maximum when $q = q^*_R = q^*_{SC} = q^*$ and $\omega = \omega^*$ for the considered case of the triangular distribution of the random variable ξ.

Write the expected profit of the supplier $E[\pi_S(r, t)]$ for obtained $q = q^*_R = q^*_{SC} = q^*$ and $\omega = \omega^*$.

$$E\left[\pi_S\ (r, t)\right] = \left(\omega^* - c_S\right) q^* - r \left(\frac{h}{3} - \frac{\left(h - q^*\right)^3}{3h^2}\right) + tr.$$

Using expressions (26) and (27), transform the function of the supplier's expected profit:

$$E\left[\pi_S\ (r, t)\right] = rh \left(\frac{c - v}{p - v}\left(1 - \sqrt{1 - \frac{p - c}{p - v}}\right) - \frac{1}{3} + \frac{\left(\sqrt{1 - \frac{p - c}{p - v}}\right)^3}{3}\right) + rt.$$

The verification of the sufficient extremum condition for the function $E\left[\pi_S\ (r,t)\right]$ is carried out using the Hessian matrix:

$$G=\begin{pmatrix} \frac{\partial^2 E[\pi_S\ (r,t)]}{\partial r^2} & \frac{\partial^2 E[\pi_S\ (r,t)]}{\partial r \partial t} \\ \frac{\partial^2 E[\pi_S\ (r,t)]}{\partial t \partial r} & \frac{\partial^2 E[\pi_S\ (r,t)]}{\partial t^2} \end{pmatrix}.$$

It is necessary to calculate all partial derivatives of the first and second order of the function $E\left[\pi_S\ (r,t)\right]$ for r and t:

$$\frac{\partial E\left[\pi_S\ (r,t)\right]}{\partial r}=h\left(\frac{c-v}{p-v}\left(1-\sqrt{1-\frac{p-c}{p-v}}\right)-\frac{1}{3}+\frac{\left(\sqrt{1-\frac{p-c}{p-v}}\right)^3}{3}\right)+t.$$

$$\frac{\partial E\left[\pi_S\ (r,t)\right]}{\partial t}=r,$$

$$\frac{\partial^2 E\left[\pi_S\ (r,t)\right]}{\partial r^2}=0,$$

$$\frac{\partial^2 E\left[\pi_S\ (r,t)\right]}{\partial t^2}=0,$$

$$\frac{\partial^2 E\left[\pi_S\ (r,t)\right]}{\partial r \partial t}=\frac{\partial^2 E\left[\pi_S\ (r,t)\right]}{\partial t \partial r}=1.$$

To check the sufficient condition, it is necessary to calculate the determinant of the Hessian matrix:

$$\Delta=\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}=-1\ .$$

The negative value of the determinant of the Hessian matrix allows us to conclude that there are no extremes of the expected supplier's profit function $E\left[\pi_S\ (r,t)\right]$, and therefore, there is no maximum of this function. From this fact it follows that the sales rebate contract does not fulfill the individual rationality property for the supplier, which means it is not coordinating for the considered case of the triangular distribution of the random variable ξ.

Now check whether the sales rebate contract conditionally coordinate the supply chain for the case of the triangular distribution of the random variable ξ. Find the values of r and t at which the expected profit of the supplier $E[\pi_S(r,t)]$ for obtained ω^* and q^* is at least not less than by the wholesale price contract with the same values of ω^* and q^*.

The expected profit of the supplier $E[\pi_S(r,t)]$ for obtained ω^* and q^* has the following form:

$$E\left[\pi_S\ (r,t)\right]=\left(\omega^*-c_S\right)q^*-r\left(\frac{h}{3}-\frac{\left(h-q^*\right)^3}{3h^2}\right)+tr=$$
$$=\frac{c-v}{p-v}rq^*+r\left(\frac{\left(h-q^*\right)^3}{3h^2}-\frac{h}{3}+t\right). \quad (28)$$

If the retailer is unable to sell the volume exceeding the threshold t specified by the contract, then the regulation of the supplier and the retailer's actions takes place as part of the wholesale price contract. Substitute the expression (27) for the wholesale price ω^* into the expression (6) and determine the expected profit of the supplier $E[\pi_S^{wholesale}]$ in case when he does not pay any rebate to the retailer:

$$E\left[\pi_S^{wholesale}\right] = (\omega^* - c_S)\, q^* = \frac{c-v}{p-v} r q^*.$$

The expression (28) for $E\left[\pi_S\ (r,t)\right]$ can be represented as follows:

$$E\left[\pi_S\ (r,t)\right] = E\left[\pi_S^{wholesale}\right] + r\left(\frac{(h-q^*)^3}{3h^2} - \frac{h}{3} + t\right). \qquad (29)$$

Let us analyze the expression for $E\left[\pi_S\ (r,t)\right]$. In the case of the triangular distribution of the random variable ξ, the supplier should set a threshold sales volume t such that:

$$t > \frac{h}{3} - \frac{(h-q^*)^3}{3h^2}.$$

If the supplier sets $t > \frac{h}{3} - \frac{(h-q^*)^3}{3h^2}$ as well as the wholesale price $\omega = \omega^*$, then such a contract is profitable for him, since the expected profit of the supplier is more than without the use of the rebate ($E\left[\pi_S\ (r,t)\right] > E\left[\pi_S^{wholesale}\right]$). The retailer in this case chooses the purchase volume $q = q^*$, thereby maximizing her expected profit and the expected profit of the supply chain. Under such conditions, the sales rebate contract is beneficial for both the retailer and the supplier, and will also ensure the achievement of conditional supply chain coordination.

We show how the expected profit of the supply chain is distributed between the supplier and the retailer at $t > \frac{h}{3} - \frac{(h-q^*)^3}{3h^2}$ in the case when the supplier pays the rebate to the retailer and when the interaction occurs within the framework of the wholesale price contract. Let us find the expected profits of the supplier and retailer for obtained $q = q^*$ and $\omega = \omega^*$ in the situation when the rebate is not paid. We substitute the expression (27) for the wholesale price ω^* into the expressions for the expected profits of the supplier and the retailer (5) and (6).

$$E\left[\pi_R^{wholesale}\right] = (p-v)\left(\frac{h}{3} - \frac{(h-q^*)^3}{3h^2}\right) - \left(\frac{c-v}{p-v} r + c - v\right) q^*,$$

$$E\left[\pi_S^{wholesale}\right] = \frac{c-v}{p-v} r q^*.$$

Find the expected profit of the retailer $E\left[\pi_R^{rebate}\right]$ for obtained $q = q^*$ and $\omega = \omega^*$ in the case of the rebate payout. To do this, we substitute the expression (27) for the wholesale price ω^* into expression (11).

$$E\left[\pi_R^{rebate}\right] = (p-v+r)\left(\frac{h}{3} - \frac{(h-q^*)^3}{3h^2}\right) - \left(\frac{c-v}{p-v} r + c - v\right) q^* - tr.$$

This expression can also be represented as follows:

$$E\left[\pi_R^{rebate}\right] = E\left[\pi_R^{wholesale}\right] - r\left(\frac{(h-q^*)^3}{3h^2} - \frac{h}{3} + t\right).$$

The expected profit of the supplier $E\left[\pi_S^{rebate}\right]$ for obtained $q = q^*$ and $\omega = \omega^*$in the case of the rebate payout has the form (expression 29):

$$E\left[\pi_S^{rebate}\right] = E\left[\pi_S^{wholesale}\right] + r\left(\frac{(h-q^*)^3}{3h^2} - \frac{h}{3} + t\right).$$

Thus, at $t > \frac{h}{3} - \frac{(h-q^*)^3}{3h^2}$ $E\left[\pi_S^{rebate}\right] > E\left[\pi_S^{wholesale}\right], E\left[\pi_R^{rebate}\right] < E\left[\pi_R^{wholesale}\right]$.

Note that the initial terms of the sales rebate contract suggest that the supplier gives part of his expected profit to the retailer in the form of the paid rebate. However, as we show under the restriction on parameter t, by offering such a rebate, the supplier increases his expected profit by reducing the expected retailer's profit. This happens because the rebate size is taken into account in the offered wholesale price: the higher the rebate size, the higher the requested wholesale price is.

5. Constructing a coordinating sales rebate contract under the assumption that the rebate defined as a percentage of the retail sale price (p)

Let us now consider the solution of the supply chain coordination problem using the example of pharmaceutical company engaged in the delivery of specific products to medical institutions in the north-west market. More than 90% of products are purchased from a foreign supplier – a big international manufacturer. The retailer company is the exclusive partner of the manufacturer, that is, it is the only company that works directly with the supplier. The company's clients are both private companies and government institutions. The interaction with them occurs either directly or through additional intermediaries – dealers. It is worth noting, that in the case of selling products through dealer companies, the retailer company tracks the movement of products to the final consumer.

Sales of products to government agencies occur through the conclusion of state procurement contracts; sales to private companies occur through the conclusion of the wholesale contracts or the contracts with discounts depending on the volume purchased. Interaction with dealers is also organized through the wholesale price contracts and the contracts with discounts depending on the volume purchased. In 2016/2017 the interaction of the retailer company with the supplier was regulated by the wholesale price contract. In 2018 the supplier decided to switch to the sales rebate contract. Note that the supplier has much more market power than the retailer company, since his products are unique. It is he who offers the retailer the terms of the contract.

The solution of the theoretical problem of supply chain coordination is given for two supply chain partners - the supplier and the retailer - so we will consider the interaction between the manufacturing company (supplier) and the retailer company. The contract concluded between these companies is essentially a contract for the supply of products between economic entities, which is concluded according to certain rules and contains mandatory sections. The considered task of supply chain coordination, that is, determination the parameters of a coordinating contract, is related to the commercial terms of such a supply contract, which affect the final financial results of the companies, that is, the sizes of their profits. In other words, the task is to determine the parameters of the coordinating contract, which theoretically can be stated in the section with the commercial terms of the contract to be concluded. Note that the contract terms are determined annually for the entire range of products; to solve the problem, we assume that the contract is for one type of product (one vendor code).

Consider in the more detailed form how the procurement process in the supply chain is organized. The purchase occurs every week. On Friday the retailer makes orders for the supply of products and sends these data to the supplier. The products arrive at the retailer's warehouse on Tuesday and during this week the products are sold to the company's customers. Thus, in this model, the sales season is one week. Note that, as a rule, purchase orders are formed at the request of customers; however, for some types of products that are often sold, the purchase takes place with a certain safety stock.

The solution of the supply chain coordination problem begins with the assumption about the law of demand distribution for a specific type of product. In the previous chapter the task of constructing a conditionally coordinating sales rebate contract is solved under the assumption that the demand for the product has the triangular distribution. However, in practice, obtaining information on the parameters of demand distribution is an intractable task. This fact significantly limits the possibility of applying the considered models to solve real management problems. The alternative option is to build an empirical distribution function not of demand, but of the sales volume. Access to such data is much easier.

As noted earlier, the procurement process in the retailer company takes place weekly, so to build an empirical distribution function of the sales volume, data on sales volume for the week is taken for several products separately. The products that are most frequently sold are analyzed, for a total of 44 products. In the sample for each product – 102 observations (2 years to 51 weeks in each year, since sales usually begin from the second week of the year). The random variable τ_i is the number of units of a certain product sold per week (measured in units), $i = B, \dots AZ$. Data for each product is analyzed for outliers and cleared of them. Based on the analysis of the constructed histograms, an assumption about the triangular distribution of the random variable τ_i $(G(h))$ is made. The chosen significance level is 0,01. To test this assumption, the following hypotheses are put forward:

$$H_0\colon \tau_i \in G(h),$$

$$H_a\colon \tau_i \notin G(h).$$

Testing the hypothesis is carried out using the Kolmogorov-Smirnov test. As a result, five products are identified, the weekly sales volume for which is subject to the triangular distribution law. For further analysis, one of five products is selected. The histogram of the random variable τ for the considered product is presented in the figure 1.

Thus, we can conclude that the weekly sales volume for the considered product is the random variable distributed according to the triangular law with the parameter $h = 61$. Note that the parameter h shows the maximum sales of the product in one week.

Consider the task of constructing a coordinating contract for the considered product in the framework of the model described earlier. To solve this problem, it is necessary to know the law of demand distribution (random variable ξ); however, from the available data it is possible to determine only the distribution law of the sales volume (random variable τ). We assume that the random variable ξ has the same

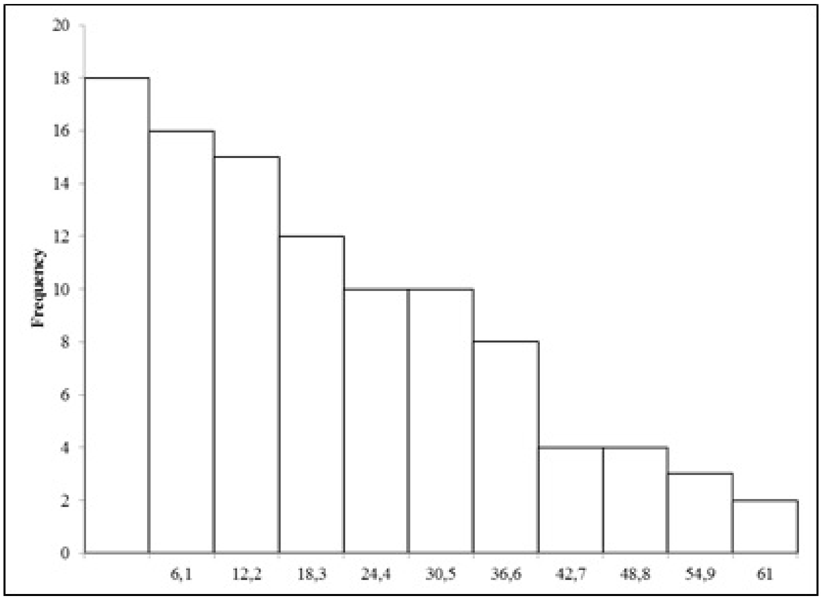

Fig. 1. The histogram of sales volume for the considered product.

distribution law. To find the parameters of the coordinating contract we will use the data presented in the table 2.

Table 2. Initial data

p	2 385 (rub.)
v	0 (rub.)
c_S	732 (rub.)
c_R	90 (rub.)
c	822 (rub.)

Note that the salvage value per unit (v) is zero, since the retailer company does not have the opportunity to sell the remaining products within the framework of one sales season. The retailer costs (c_R) include all costs associated with the sale of the production unit.

According to the results presented in the previous section, the sales rebate contract conditionally coordinates the supply chain under the certain restriction on the parameter t. We use the results obtained in the previous section to find the parameters of the conditionally coordinating contract for the considered product, the demand for which has the triangular distribution with the parameter $h = 61$.

Let us determine the optimal purchase volume (q^*), at which the expected profits of the supply chain and the retailer reaches maximum. To do this, in the expression (26) we substitute the specific value of the parameter of the demand distribution $h = 61$:

$$q^* = h\left(1 - \sqrt{1 - \frac{p - \mathrm{c}}{p - v}}\right) = 61\left(1 - \sqrt{1 - \frac{2\ 385 - 822}{2\ 385}}\right) = 25,19 \approx 25.$$

Since the volume of purchased products (measured in units) can only be an integer, it is necessary to round the result to an integer value.

Define the threshold sales volume (t), above which the sales rebate contract is beneficial for the supplier. As is shown in the previous section, this sales volume must be greater than the expected sales volume $E[\tau]$. In order to find the limit on t, we substitute the specific value of the demand distribution parameter $h = 61$ into the expression (15) and the found value of the optimal purchase volume $q^* = 25,19$:

$$E[\tau] = \frac{h}{3} - \frac{(h - q^*)^3}{3h^2} = \frac{61}{3} - \frac{(61 - 25)^3}{3 * 61^2} \approx 16.$$

Rounding occurs from the grounds that the sales volume can only be an integer. Thus, the sales rebate contract for the considered product is beneficial for the supplier with $t > 16$. Note that to ensure that the regulation of supply chain partners interactions occur precisely within the framework of the sales rebate contract, the retailer must order a larger volume than the threshold level of sales, $q^* > t$. Otherwise, the rebate is not paid and it is a wholesale price contract. Thus, we can write the following inequality for the parameter t: $E[\tau] < t < q^*$; and for the considered example – $16 < t < 25$.

To determine the next parameter of the coordinating contract – the wholesale price ω^*, it is necessary to determine the size of the rebate (r). However, according to the results of the previous section, it is not possible to obtain any restrictions on the parameter r (rebate size). This parameter is selected by the supplier and responsible for how the expected supply chain profit is distributed among the supply chain partners. Consequently, the supplier chooses such a value of r at which his expected profit is maximum, and the expected retailer's profit is acceptable for her (not less than a certain exogenously established level – reservation profit). To solve this problem, we assume that the rebate size (r) is defined as a certain percentage of the retail price (p), that is, $r = \delta p$, where $\delta \in [0, 1]$. This approach is used in (Chiu et al., 2012), and is also used by real companies to determine the parameter r (Chiu et al., 2011-a). Since δ can take a set of values on the interval $[0, 1]$, there is also a set of values of the parameter r, and hence the set of solutions of the model, which conventionally coordinate the supply chain. In addition, the parameter t can also take different integer values on the interval $(16,\ 25)$.

To find all possible solutions and calculate the expected profits for the supplier, the retailer and supply chain, it is better to build a table in Excel. Consider an example of calculating the remaining parameters of the coordinating contract and the expected profits of the retailer, the supplier and the supply chain at $\delta = 0,01$, $t = 17$.

Firstly, determine the size of the rebate (r):

$$r = \delta p = 0,01 * 2\ 385 = 23,85.$$

Then, to determine the wholesale price ω^*, we substitute the obtained value for r into expression (27):

$$\omega^* = c_S + \frac{c - v}{p - v} r = 732 + \frac{822}{2\ 385} * 23,85 = 740,22.$$

Thus, the parameters of the coordinating contract are presented in the table 3.

Table 3. The parameters of conditionally coordinating sales rebate contract

q^*	25 (pcs.)
ω^*	740,22 (rub./pcs.)
r	23,85 (rub./pcs.)
t	17 (pcs.)

Find the expected profits of the retailer, the supplier and the supply chain with the found parameters of conditionally coordinating contract. The values from tables 2 and 43, as well as the value of the parameter of the demand distribution $h = 61$, are substituted into expressions (11)–(13). Calculate the values of expected profits:

$$E\left[\pi_R\right] = 17\ 751,16,$$

$$E\left[\pi_S\right] = 225,68,$$

$$\mathrm{E}[\Pi_{SC}] = 17\ 976,84.$$

In addition, we find the supplier's profit in the absence of any rebate, that is, if the interaction of the supply chain partners is regulated by the wholesale price contract for obtained ω^* and $q = q^*$. To do this, we substitute the necessary parameters in the expression (6):

$$E\left[\pi_S^{wholesale}\right] = \left(\omega^* - c_S\right) q^* = 205,5.$$

The following conclusions can be drawn from the analysis of these expressions. The expected profits of the supply chain and the retailer are maximum, since the volume of purchased products is $q = q^*$. The expected profit of the supplier in the sales rebate contract is higher than in the wholesale price contract. However, 98,7% of the expected supply chain profit of the chain is "taken away" by the retailer, and only 1,3% goes to the supplier. Thus, despite the fact that the contract with these parameters is conditionally coordinating, it will probably not be concluded in practice, since the supplier receives the negligible profit compared with the retailer. In addition, the supplier is the leader in the game, so it is likely that such conditions are not acceptable to him.

In Excel solutions for the task of supply chain coordination when the parameter δ changes from 0,01 to 0,455 with a step of 0,005 and for different values of the parameter t belonging to the interval (16, 25) are found. The expected profits of the retailer, the supplier and the supply chain are calculated, as well as the percentage ratio in which the expected supply chain profit is distributed among partners involved. Note that the expected profit of the supply chain does not change when the parameters r and t change, since these parameters affect only the distribution of this profit between the supplier and the retailer. If the rebate size increases, leaving the other contract parameters (except the wholesale price ω^*, which functionally depends on the rebate size r) unchanged, the share of the supplier's expected profit in the expected supply chain profit increases, and the share of the retailer decreases. With an increase in the sales threshold t and fixed values of other parameters, the share of the expected profit of the supplier also increases.

Under the restrictions on the parameter t according to the previous section, the expected profit of the supplier in the case of the sales rebate contract should be higher than in the case when such a rebate is not paid. For each obtained wholesale price value ω^*, we calculate the expected profit of the supplier in the case of the wholesale price contract for obtained $q = q^*$. As expected, in the case of the sales rebate contract, the expected profit of the supplier is higher than for the wholesale price contract. Thus, the sales rebate contract really conditionally coordinates the supply chain.

Based on the calculations, we construct graphs of the expected profits of the supplier and the retailer depending on the size of the rebate, namely on the parameter δ, for all possible values of the parameter t belonging to the interval $(16,\ 25)$. On each graph, we also construct the dependence of the expected profit of the supplier in the absence of the rebate, that is, if the interaction of the supply chain partners is regulated by wholesale price ω^* contract, on the parameter δ. Figures 2 and 3 show graphs of the expected profits of the supplier and the retailer on the rebate size for $t = 17$ and $t = 24$.

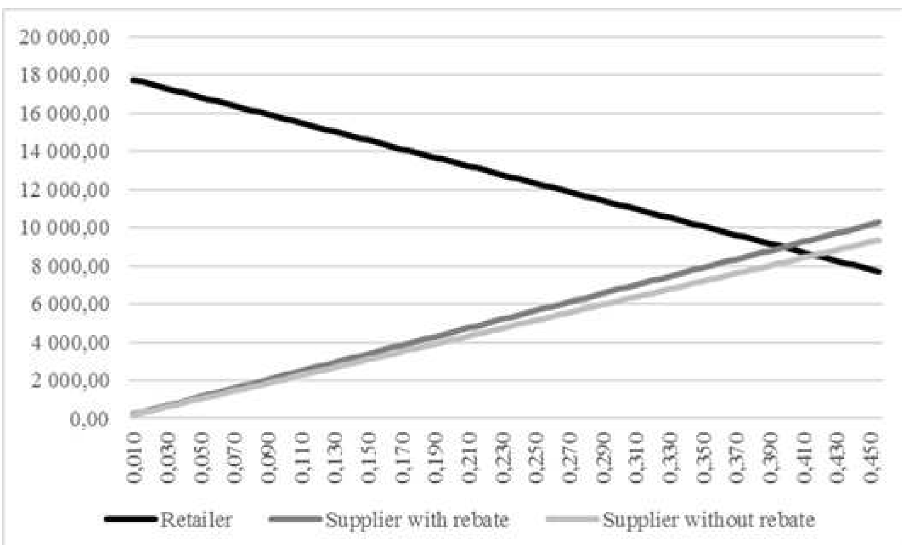

Fig. 2. The expected profits of the supplier and the retailer for $t = 17$.

The graphs clearly demonstrate that the function of the supplier's expected profit does not have a maximum, as it infinitely increases in r and t. That is why, as it is shown in the previous section, there is no solution that unconditionally coordinates the supply chain in the case of the sales rebate contract. Note also that the smaller the value of the threshold sales volume (t), the more gentle the graphs of the functions of the expected profits of the retailer and the supplier are. It can be concluded that the larger the threshold sales volume (t), the faster with an increase in the size of the rebate paid, the retailer's share in the expected supply chain profits decreases and the supplier's share increases, respectively. For example, at $t = 17$, the expected retailer's profit equals the expected supplier's profit for the considered product, with the rebate amount (r) of 924, 08 rubles. ($\delta = 0,395$). When $t = 21$, this situation is already achieved when $r = 679,73$ rubles ($\delta = 0,285$); for $t = 24$ – at $r = 548,55$ rubles ($\delta = 0,23$). Note also that for $t = 17$, the retailer's expected

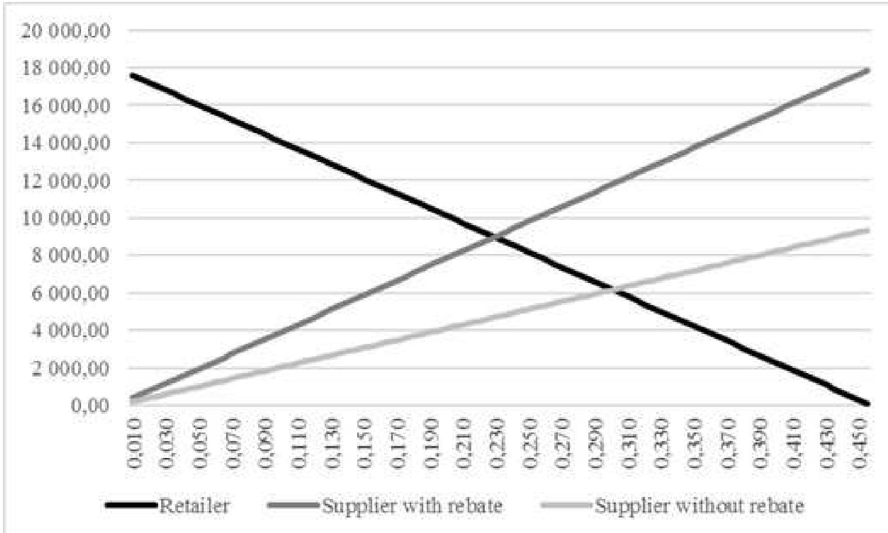

Fig. 3. The expected profits of the supplier and the retailer for $t = 24$.

profit becomes negative when $r > 1\ 896,08$ rubles ($\delta > 0,795$); for $t = 21$ when $r > 1\ 323,68$ rubles ($\delta > 0,555$); for $t = 24$ when $r > 1\ 085,18$ rubles ($\delta > 0,455$). Hence, we can conclude that if the supplier chooses the threshold sales volume close to the optimal purchase volume (q^*), then the number of possible options for the rebate size is less than when the threshold sales volume is close to the expected sales volume ($E[\tau]$). Moreover, the larger the value of the threshold sales volume, the smaller the maximum possible rebate value (the value of the parameter r and, accordingly, δ) at which the expected profits of both supply chain partners are non-negative.

If we compare the graphs of the supplier's expected profit in the case when the rebate is paid and when there is the wholesale price contract (without the rebate), then we notice that the graph of the expected supplier's profit in the case of the wholesale price ω^* contract is even more gentle. For any value of the parameter t, the graph of the function of the supplier's expected profit in the wholesale price contract is lower than in the case of the rebate payment. Thus, with the same wholesale price values, the supplier gets a lower expected profit for the wholesale price contract than for the sales rebate contract. This once again proves the possibility of the sales rebate contract to conditionally coordinate the supply chain.

6. Constructing a coordinating sales rebate contract under the assumption that the rebate is defined as a percentage of the wholesale price (ω)

After studying the sales rebate contract, which is actually concluded between the retailer company and the manufacturing company in 2018, it was found that the rebate amount (r) is defined as a certain percentage of the wholesale price (ω). At the same time, the rebate amount for the considered product category, which includes the product being analyzed, may be 2%, 4%, 6% or 8% of the wholesale price. This approach is also used in other real-life cases described in (Chiu et al., 2011-a).

Let us build solutions of the supply chain coordination problem using the sales rebate contract for the considered product in Excel, assuming that the rebate size is now determined as a percentage of the wholesale price, that is, $r = \gamma\omega$, where $\gamma \in [0, 1]$, and the parameter t takes different integer values on the interval $(16,\ 25)$. Using the previously obtained value for the optimal purchase volume (q^*), we consider an example of calculating the remaining parameters of the coordinating contract and the expected profits of the retailer, the supplier and the supply chain for $\gamma = 0,01$, $t = 17$.

If earlier it was necessary to firstly determine the size of the rebate (r), and then calculate the optimal wholesale price (ω^*), then now we first determine the value of the wholesale price. To do this, we substitute the assumption about the method of determining the size of the rebate in the expression (27):

$$\omega^* = c_S + \frac{c-v}{p-v} r = c_S + \frac{c-v}{p-v}\gamma\omega^*.$$

Hence

$$\omega^* = \frac{c_S}{1 - \frac{c-v}{p-v}\gamma}.$$

For the considered example:

$$\omega^* = \frac{732}{1 - \frac{822}{2\ 385} * 0,01} = 734,53.$$

The rebate size:

$$r = 0,01 * 734,53 = 7,35.$$

Thus, the parameters of the coordinating contract are presented in the table 4. Find

Table 4. The parameters of conditionally coordinating sales rebate contract

q^*	25 (pcs.)
ω^*	734,53 (rub./pcs.)
r	7,35 (rub./pcs.)
t	17 (pcs.)

the expected profits of the retailer, the supplier and supply chain with the found parameters of conditionally coordinating contract. For this, the values from tables 2 and 4, as well as the value of the parameter of the demand distribution $h = 61$, are substituted into expressions (11)–(13).

$$E\left[\pi_R\right] = (p - v + r)\left(\frac{h}{3} - \frac{(h-q^*)^3}{3h^2}\right) - (\omega^* + c_R - v)\, q^* - tr = 17\ 907,34.$$

$$E\left[\pi_S\right] = (\omega^* - c_S)\, q - r\left(\frac{h}{3} - \frac{(h-q^*)^3}{3h^2}\right) + tr = 69,51.$$

$$\mathrm{E}[\Pi_{SC}] = (p - v)\left(\frac{h}{3} - \frac{(h-q^*)^3}{3h^2}\right) - (c - v)\, q^* = 17\ 976,84.$$

In addition, we find the supplier's profit in the absence of any rebate, that is, if the interaction of the supply chain partners is regulated by the wholesale price contract for obtained ω^* and $q = q^*$. To do this, we substitute the necessary parameters in the expression (6):

$$E\left[\pi_S^{wholesale}\right] = (\omega^* - c_S)\, q^* = 63,29.$$

Let us analyze the results. The expected profits of the supply chain and the retailer are maximum, since the purchase volume is $q = q^*$. The expected profit of the supplier for the sales rebate contract is higher than for the wholesale price ω^*contract. However, as in the previous case, the retailer "takes away" most of the supply chain expected profit (99,3%), and only 0,7% of the expected profit goes to the supplier, that probably does not agree with such conditions. Note that when comparing two approaches to calculating the size of the rebate, the wholesale price values are comparable, while for the first approach the size of the rebate is more than 3 times higher than for the second.

In Excel solutions for the task of supply chain coordination when the parameter γ changes from 0,01 to 0,985 with a step of 0,005 and for different values of the parameter t belonging to the interval $(16,\ 25)$ are found. The expected profits of the retailer, the supplier and supply chain, the percentage ratio in which the supply chain expected profit is distributed among partners, and the expected profit of the supplier in the case of the wholesale price ω^* contract for obtained $q = q^*$ are calculated. Figures 4 and 5 show graphs of the expected profits of the supplier and the retailer in the case of the rebate payment and the expected profit of the supplier in the wholesale price contract on the rebate size, namely the parameter γ, for $t = 17$ and $t = 24$. Note that all conclusions made under the assumption that the rebate is determined as a percentage of the retail price are fair for the case under consideration when the rebate amount is a certain percentage of the wholesale price. The sales rebate contract really conditionally coordinates the supply chain.

If we compare the two approaches to determining the rebate size, we can conclude that for the first approach, as the size of the paid rebate increases, the share of expected supply chain profits that the retailer receives decreases faster than for the second approach, and the share of the expected supply chain profit that the supplier receives increases respectively. This is due to the fact that when using the second approach, the rebate paid is lower than for the first approach, since initially the percentage is determined from the lower value $(\omega < p)$.

Based on the analysis of the constructed solutions in the task of supply chain coordination for the considered product, the demand for which has the triangular distribution with the parameter $h = 61$, we can draw the following conclusions. First, while testing the mathematical model on real data, it is possible to obtain such parameters of the coordinating sales rebate contract, for which:

1. The expected supply chain profit is maximum;
2. The expected retailer's profit is maximum;
3. The expected supplier's profit is higher than in the case of the wholesale price contract.

Secondly, in the task of conditional coordination for a specific type of product, under the proposed assumption about the determination of the rebate size, a number of solutions are found, that is, a set of contract parameters, that allow the

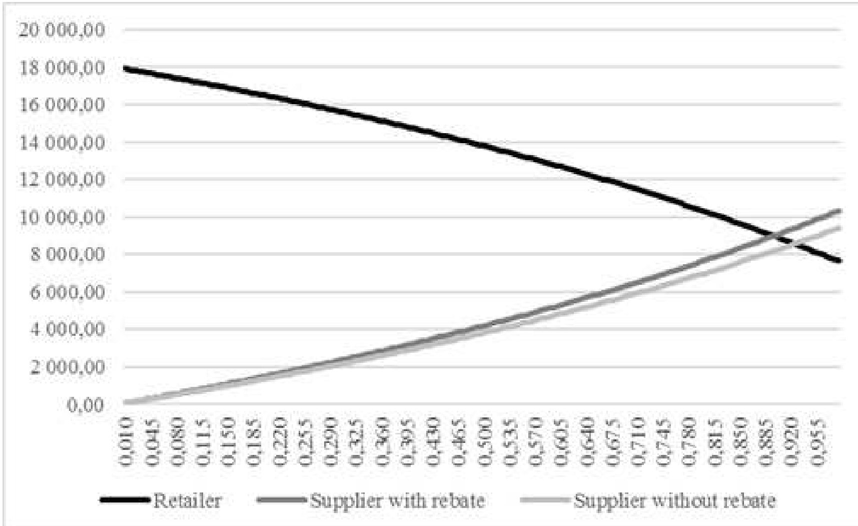

Fig. 4. The expected profits of the supplier and the retailer for $t = 17$.

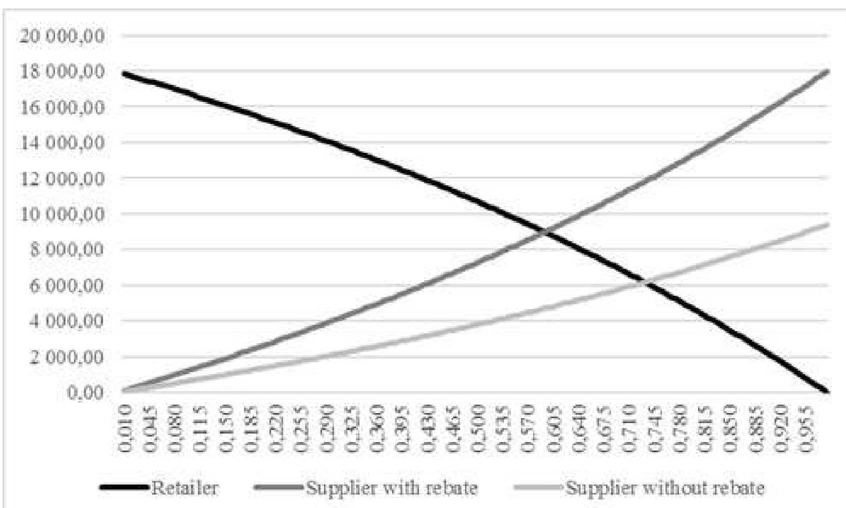

Fig. 5. The expected profits of the supplier and the retailer for $t = 24$.

supply chain to be conditionally coordinated. In other words, in the model under consideration there are more than one solution.

Finally, the constructed model of a coordinating contract can be an effective tool for making management decisions related to the interaction of partners involved in the supply chain, both for the supplier (international manufacturing company) and for the retailer company. For a model in which the supplier is the leader and offers the retailer contract terms, there are many solutions in which there is a different distribution of the maximum expected supply chain profit among the partners involved in the supply chain. While making decisions about the specific parameters of the sales rebate contract, the supplier, using this model, can analyze what part of the expected profit of the supply chain he gives to the retailer and which part he retains. At the same time, it is obvious that at extreme values of the contract parameters, when the share of the supply chain expected profit of one partner can be less than 1%, the contract will not be concluded. Thus, using this model, the supplier can determine the desired ratio of his expected profit to the retailer's expected profit and, based on this, make further decisions about the parameters of the contract. Here we denote that the expected profit of each supply chain partner must be higher than a certain exogenously established level (reservation profit). The parties understand their alternatives for the use of resources, so even if the supplier is the leader and offers the retailer the terms of the contract, he must in any case ensure that the retailer's expected profit from participation in the contract will be higher than this established level. The retailer can also analyze the distribution of the expected supply chain profit among the supply chain partners and determine if the supplier's proposed contract parameters are acceptable, that is, if she receives more expected profit than a certain exogenously determined level (reservation profit).

In general, we can conclude that the sales rebate contract is a flexible mechanism of supply chain coordination, since it allows to choose parameters at which the expected supply chain profit is maximum, and any distribution of this profit between supply chain partners is also possible. The decision about what percentage of the maximum possible expected supply chain profit goes to the retailer and what percentage goes to the supplier, in practice, must be made as a part of the negotiation process.

7. Conclusion

The paper considers the sales rebate contract, its features and application in practice. The simple model for risk-neutral supply chain partners is presented, and the supply chain coordination problem solving is proposed. The solution of this problem is given under the assumption of triangular distributed demand. It was shown that the sales rebate contract is not coordinating, as it does not provide the fulfillment of the condition of individual rationality for the supplier (the function of the expected profit of the supplier does not have maximum points). However, such a contract allows achieving conditional coordination of the supply chain, when the expected profits of the supply chain and the retailer are maximum, and the expected profit of the supplier is greater than for the case of the wholesale price contract. It can be argued that the use of sales-rebate contract under certain conditions is beneficial for both supply chain partners in the supply chain and allows to maximize the expected profit of the chain. Thus, it was approved that the problem of supply chain profit maximization can be solved using the sales rebate contract.

To verify the algorithm for constructing a coordinating contract, the case of the pharmaceutical supply chain was investigated. The interaction between a company engaged in the supply of specific products to medical institutions which is a retailer in the considered model and a large international manufacturer which is a supplier in the considered model was explored. As a result of the study of the retailer's data on weekly sales, it was found that there are products, the demand for which is a random variable that has a triangular distribution. Based on this information and using expressions for the parameters of the coordinating contract obtained earlier, solutions were found in the supply chain coordination problem under the assumption that the supplier determines the amount of the rebate paid as a percentage of the retail price and as a percentage of the wholesale price. The sales rebate contract was also compared to the contract at wholesale price and it was shown that the supplier always gets more expected profit under the sales rebate contract than under the contract at wholesale price. Thus, the possibility of achieving conditional coordination and obtaining the maximum of expected profit of the supply chain using sales rebate contract under the proposed assumptions was demonstrated on real data.

The analysis of the solutions also showed the effectiveness of the sales rebate contract in terms of the distribution of the supply chain expected profit between partners. Under the proposed assumptions, the found parameters of the coordinating contract allow achieving any distribution of the supply chain expected profit between the supplier and the retailer. This means that the sales-rebate contract act as a coordination mechanism not only in the model, where the supplier is a leader and offers the retailer a take-it-or-leave-it contract, but also in more complex models, where the contract parameters are determined within the negotiation process. Using the proposed model, both the supplier and the retailer can observe how the expected profit of the supply chain is redistributed between them depending on the specific values of the contract parameters. In addition, since each partner knows the minimum amount of expected profit that he expects to receive from participation in the contract, it is possible to impose additional restrictions on the values of the parameters. Thus, the proposed model of finding a solution to the problem of supply chain coordination can serve as an effective management tool in making decisions about the choice of the contract parameters.

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