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Two-Level Cooperative Game on Hypergraph*

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Abstract In the paper, the cooperative game with a hypergraph communication structure is considered. For this class of games, a new allocation rule was proposed by splitting the original game into a game between hyperlinks and games within them. The communication possibilities are described by the hypergraph in which the nodes are players and hyperlinks are the communicating subgroups of players. The game between hyperlinks and between players in each hyperlink is described. The payoff of each player is influenced by the actions of other players dependent on the distance between them on hypergraph. Constructed characteristic functions based on cooperative behaviour satisfy the convexity property. The results are shown by the example.

Keywords: cooperation, characteristic function, hypergraph, communication structure.

1. Introduction

In classic cooperative games with transferable utility, it is assumed that all players have an opportunity to form a grand coalition and the total payoff from the cooperation can be distributed among the players.

In general cases, not all players can communicate with each other due to some economic, transport and other restrictions. To define the possibilities of communication the communication structure can be used. The undirected graph can represent a communication structure. In the graph, only nodes that have an edge between them can interact. Such class of games was initially studied by Myerson (Myerson, 1977). After that, games with communication structure have received attention in cooperative game theory. Owen (Owen, 1986) studied games with a tree as a communication structure. Meessen (Meessen, 1988) introduced the positional value for the games with graph communication structure.

The TU-games with hypergraph communication structure were studied by Nouweland, Borm and Tijs (Nouweland et al., 1992). They characterized the Myerson value and the positional value for these games. The third value, which is called degree value for games with a hypergraph communication structure, was introduced in (Shan et al., 2018).

In the paper cooperative games with hypergraph communication structure are considered. We propose two-level based approach for these games. The paper is organized as follows. First, basic definitions and notations concerning the hypergraph communication structure are given. Then the definition of the game with discounted payoff function is provided. Next, described two-level based approach and subgames between and inside hyperlinks are considered. Finally, the results are explained on an example.

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2. Communication Structure

In the paper, we consider a game with a hypergraph communication structure. Hypergraph is a generalization of a graph in which an edge can join any number of vertices. In contrast, in an ordinary graph, an edge connects exactly two vertexes. Formally hypergraph is a pair $(N, \mathscr{H})$, $\mathscr{H} \subseteq \{H \in 2^N \big| |H| \geqslant 2\}$, where N is a finite set of nodes and $\mathscr{H}$ is a given set of subsets of N, each element in $\mathscr{H}$ is called hyperlink. Notice that the cardinality $|H|$ should be greater or equal to two. If all hyperlinks include only 2 nodes the hypergraph can be considered as a graph. A simple example of a hypergraph is presented in Fig.1.

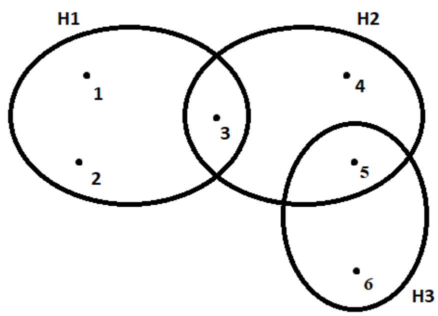

Fig. 1. Example of Hypergraph

Reduction of hypergraph $(N, \mathscr{H})$ can be obtained by removing all hyperlinks which are entirely included in other hyperlinks from the original hypergraph $(N, \mathscr{H}')$. If a hypergraph coincides with its reduction, that is, it does not have hyperlinks inside others, such hypergraph is called reduced.

The path between two hyperlinks H_i and H_j is a sequence of hyperlinks $H_{k_1}, H_{k_2}, \ldots, H_{k_l}$ where:

$$1) H_i = H_{k_1}, H_j = H_{k_l}$$

$$2) \forall i : 1 \leq i \leq k-1, H_{l_i} \cap H_{l_{i+1}} \neq \emptyset.$$

The minimal path between two hyperlinks is the path that contains minimal number of hyperlinks.

The distance between two vertexes i, j is defined as the number of hyperlinks l in the minimal path between hyperlinks H_{k_1}, H_{k_l} where $i \in H_{k_1}, j \in H_{k_l}$

Two vertexes that have a path between them are called connected. Define neighbours with level k for each node.

Neighbours of vertex $i \in N$ with level 1 are vertexes $j \in N_i^1$ that are at distance 1 from i. Neighbours of vertex $i \in N$ with level k are vertexes $j \in N_i^k$ that are at distance k from i, etc. Finally, the vertexes which have no connection with i are denoted by N_i^{-1}. For any $i \in N$ the set of vertexes is split into neighbors with levels

$k \in [1, \ldots, l_i, -1]$ denoted by N_i^k where l_i is the maximal distance of connected vertex from i.

3. Definition of the game

In the game setting, the set of vertexes N is identified with the set of players. Thus $N := \{1, \ldots, n\}$ is considered as set of players. The communication structure is described by a reduced hypergraph.

Consider that each player i choose the strategy $\mathfrak{u}_i^j$ to play with connected player j from a given set of strategies $\mathfrak{U}_i^J$. Suppose that $\mathfrak{U}_i^J$ is a finite set of strategies. The set of players connected with player i denote as N_i. Denote vector the strategies of player i for all connected players as $\mathfrak{u}_i = (\mathfrak{u}_i^{m_1}, \ldots, \mathfrak{u}_i^{m_k})$ where $k = |N_i|, m_j \in N_i$. Also denote $\mathfrak{u} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_n)$.

Consider the communication between all connected players. It is natural to suppose that the distance between players should influence the payoffs. Define the discount parameter $\delta \in (0; 1)$.

Denote by l_i the distance between player i and the farthest connected player.

The payoff function for each player i is defined in the following way:

$$K_i(\mathfrak{u}_\mathrm{i}) = \sum_{m=1}^{l_i} \sum_{j \in N_i^m} \delta^{m-1} h_i^j(\mathfrak{u}_i^j, \mathfrak{u}_j^i), \quad \delta \in (0, 1), \quad h_i^j(\mathfrak{u}_i, \mathfrak{u}_j) \geq 0, i, j \in N.$$

Where $h_i^j(\mathfrak{u}_i, \mathfrak{u}_j)$ is the payoff from pairwise interaction between player i and j.

4. Cooperation

Now consider the cooperative game, that is, players agree to choose their strategies $\hat{\mathfrak{u}}_i \quad \forall i \in N$ to maximize the sum of payoffs.

$$\max_{\mathfrak{u}} \sum_{i \in N} K_i(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) = \sum_{i \in N} K_i(\hat{\mathfrak{u}}_1, \ldots, \hat{\mathfrak{u}}_n),$$

$\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_1, \ldots, \hat{\mathfrak{u}}_n)$ is a cooperative behaviour.

Characteristic function $v(S)$ is a real-valued function $v : 2^N \to R$ and $v(\emptyset) = 0$. In the classical cooperative game theory the characteristic function is defined by (Von Neumann and Morgenstern, 1994):

$$v(S) = \max_{\mathfrak{u}_S} \min_{\mathfrak{u}_{N \setminus S}} \sum_{i \in S} K_i(\mathfrak{u}_1, \ldots, \mathfrak{u}_n), \quad S \subseteq N, \tag{1}$$

here $\mathfrak{u}_S = \{\mathfrak{u}_i\}, i \in S$ and $\mathfrak{u}_{N \setminus S} = \{\mathfrak{u}_i\}, i \in N \setminus S$.

From definition it is follow that $v(N) = \sum_{i \in N} K_i(\hat{\mathfrak{u}}_1, \ldots, \hat{\mathfrak{u}}_n)$.

In this paper, we assume that the players which are not included in the coalition S destroy the communications with players in coalition S. In other words, after the coalition is fixed the restriction of a hypergraph is considered by eliminating all players $i \in N \setminus S$.

As a solution of the game, we propose a two-level based approach. On the first level, the hyperlinks are considered as players. The characteristic function is introduced in the game with hyperlinks as players. As a result, the maximal sum of payoffs obtained by cooperative behaviour is distributed among hyperlinks. On the second level, the cooperative game inside a hyperlink is considered, where players are nodes in this hyperlink.

4.1. Game between hyperlinks

In this section we consider the game between hyperlinks as players and call them h-players. Denote all coalitions of h-players as $\hat{S}$ and $N_{\hat{S}} = \bigcup_{H_i \in \hat{S}} H_i$.

The characteristic function is defined using cooperative behaviour. The approach is similar to one proposed in (Bulgakova and Petrosyan, 2019).

Suppose that for any coalition $\hat{S}$, the communication structure is constructed and denote by $N^k_{\hat{S},i}$, $i \in N_{\hat{S}}$ the neighbours with a connection level k in $N_{\hat{S}}$ for player i. Also let $l_i^{\hat{S}}$ be a distance between player i and the farthest player in $N_{\hat{S}}$.

Define the characteristic function V for the game with h-players in the following way:

$$V(\emptyset) = 0,$$

$$V(\{H_i\}) = \sum_{i \in H_i} \sum_{j \in N_i^1 \cap H_i} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i),$$

$$V(\hat{S}) = \sum_{i \in N_{\hat{S}}} \sum_{m=1}^{l_i^{\hat{S}}} \sum_{j \in N^m_{\hat{S},i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i),$$

$$V(\mathscr{H}) = \sum_{i \in N} \sum_{m=1}^{l_i} \sum_{j \in N_i^m} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i).$$

It is interesting to note that under previous assumptions about the elimination of hyperlinks and pairwise interaction this definition of characteristic function coincides with classical maxmin characteristic function (1).

Theorem 1 (Convexity). *For any two coalitions of h-players A, B we have*

$$V(A \cup B) \geq V(B) + V(A) - V(A \cap B). \tag{2}$$

Proof (of proposition). Denote $A \cap B = C$ then $\bar{A} = A \setminus C$ and $\bar{B} = B \setminus C$.

$$v(A) = \sum_{i \in N_{\bar{A}}} \sum_{m=1}^{l_i^A} \sum_{j \in N^m_{A,i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i) + \sum_{i \in C} \sum_{m=1}^{l_i^A} \sum_{j \in N^m_{A,i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i) =$$

$$= \sum_{i \in N_{\bar{A}}} \sum_{m=1}^{l_i^{\bar{A}}} \sum_{j \in N^m_{\hat{S},i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i) + \sum_{i \in N_{\bar{A}}} \sum_{m=1}^{l_i^C} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i) +$$

$$+ \sum_{i \in C} \sum_{m=1}^{l_i^C} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i) + \sum_{i \in C} \sum_{m=1}^{l_i^{\bar{A}}} \sum_{j \in N^m_{\bar{A},i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i),$$

$$v(B) = \sum_{i \in N_{\bar{B}}} \sum_{m=1}^{l_i^B} \sum_{j \in N^m_{B,i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i) + \sum_{i \in C} \sum_{m=1}^{l_i^B} \sum_{j \in N^m_{B,i}} \delta^{m-1} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i) =$$

$$= \sum_{i \in N_{\bar{B}}} \sum_{m=1}^{l_i^{\bar{B}}} \sum_{j \in N^m_{\hat{B},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) + \sum_{i \in N_{\bar{B}}} \sum_{m=1}^{l_i^{C}} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) +$$

$$+ \sum_{i \in C} \sum_{m=1}^{l_i^{C}} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) + \sum_{i \in C} \sum_{m=1}^{l_i^{\bar{B}}} \sum_{j \in N^m_{\hat{B},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i),$$

$$v(A \cap B) = v(C) = \sum_{i \in C} \sum_{m=1}^{l_i^{C}} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i),$$

$$v(A \cup B) = \sum_{i \in N_{\bar{A}}} \sum_{m=1}^{l_i^{\bar{A}}} \sum_{j \in N^m_{\hat{S},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) + \sum_{i \in N_{\bar{A}}} \sum_{m=1}^{l_i^{C}} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) +$$

$$+ \sum_{i \in C} \sum_{m=1}^{l_i^{C}} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) + \sum_{i \in C} \sum_{m=1}^{l_i^{\bar{A}}} \sum_{j \in N^m_{\hat{A},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) +$$

$$+ \sum_{i \in N_{\bar{B}}} \sum_{m=1}^{l_i^{\bar{B}}} \sum_{j \in N^m_{\hat{B},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) + \sum_{i \in N_{\bar{B}}} \sum_{m=1}^{l_i^{C}} \sum_{j \in N^m_{C,i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) +$$

$$+ \sum_{i \in C} \sum_{m=1}^{l_i^{\bar{B}}} \sum_{j \in N^m_{\hat{B},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) + \sum_{i \in N_{\bar{B}}} \sum_{m=1}^{l_i^{\bar{A}}} \sum_{j \in N^m_{\hat{A},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) +$$

$$+ \sum_{i \in N_{\bar{A}}} \sum_{m=1}^{l_i^{\bar{B}}} \sum_{j \in N^m_{\hat{B},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i).$$

After substituting these expressions into (2) and moving all the components to the left, we get the following inequality:

$$v(A \cup B) - v(B) - v(A) + v(A \cap B) =$$

$$= \sum_{i \in N_{\bar{B}}} \sum_{m=1}^{l_i^{\bar{A}}} \sum_{j \in N^m_{\hat{A},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) + \sum_{i \in N_{\bar{A}}} \sum_{m=1}^{l_i^{\bar{B}}} \sum_{j \in N^m_{\hat{B},i}} \delta^{m-1} h_i^j(\hat{\mathbf{u}}_i^j, \hat{\mathbf{u}}_j^i) \geq 0. \quad (3)$$

Because $h_i^j \geq 0$ then (3) holds and the theorem is proved.

The proof is quite similar to (Tur and Petrosyan, 2021) and is presented here for completeness of presentation.

To define the payoffs for each hyperlink we distribute the maximal sum proportionally to the individual values of the characteristic function of each h-player. Formally:

$$K_{H_i} = \frac{V(H_i)}{\sum_{H_j \in \mathscr{H}} V(H_j)} V(\mathscr{H}).$$

4.2. Game inside a hyperlink

On the second level, we consider the game inside each hyperlink. The characteristic function is defined in a similar way, using cooperative behaviour:

$$v(\emptyset) = 0,$$

$$v(\{i\}) = 0,$$

$$v(S) = \sum_{i\in S}\sum_{j\in S\setminus i} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i),$$

$$v(N) = \sum_{i\in N} h_i^j(\hat{\mathrm{u}}_i^j, \hat{\mathrm{u}}_j^i).$$

This characteristic function is convex, that is, the Shapley value belongs to the core. So Shapley value is used as an allocation rule on this level.

$$\phi_i^{H_i}(v) = \sum_{S\subseteq N, i\in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}(v(S) - v(S\setminus\{i\})).$$

Denote as $\mathscr{H}_i$ the hyperlinks which contain player i. The payoffs of hyperlinks obtained on the previous level are distributed proportionally to the Shapley value. Final allocation for the player i obtained by summing the payoffs from each hyperlink in which player is included is defined as:

$$T_i = \sum_{H_j\in\mathscr{H}_i} \frac{\phi_i^{H_j}(v)}{\sum\limits_{j\in H_j\setminus i}\phi_j^{H_j}(v)} K_{H_j}.$$

5. Example

Consider the cooperative game with player set $N = \{1, 2, 3, 4, 5, 6, 7\}$ $\delta = 0.5$ and hypergraph $\mathscr{H} = \{H_1 = \{1, 2\}, H_2 = \{2, 3, 4\}, H_3 = \{4, 5, 6\} H_4 = \{6, 7\}\}$ which is shown on the figure 2:

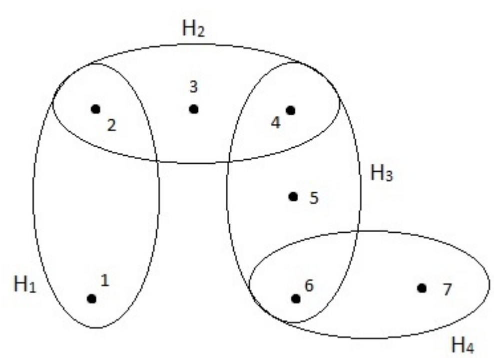

Fig. 2. Communication structure

In this example, we consider the case when each player plays a bimatrix game with players which have a connection with him. All players have the same sets of strategies $\mathfrak{u}_i = \{A, B\}$. A is the strategy to choose the first row/column and B to choose the second.

Payoffs of players 1 and 2, 1 and 3, 1 and 4, 1 and 5 are represented in the form of the following 2×2 matrices:

$$\begin{pmatrix} (11,4) & (3,15) \\ (13,9) & (20,5) \end{pmatrix} \begin{pmatrix} (3,2) & (4,9) \\ (13,1) & (7,4) \end{pmatrix} \begin{pmatrix} (13,2) & (13,6) \\ (11,11) & (3,17) \end{pmatrix} \begin{pmatrix} (4,7) & (3,9) \\ (15,3) & (7,17) \end{pmatrix}$$

Payoffs of players 1 and 6, 1 and 7, 2 and 3, 2 and 4 :

$$\begin{pmatrix} (0,2) & (6,16) \\ (2,8) & (10,8) \end{pmatrix} \begin{pmatrix} (14,9) & (12,20) \\ (7,6) & (3,20) \end{pmatrix} \begin{pmatrix} (1,20) & (16,5) \\ (8,15) & (7,3) \end{pmatrix} \begin{pmatrix} (11,11) & (10,17) \\ (14,20) & (8,14) \end{pmatrix}$$

For players 2 and 5, 2 and 6, 2 and 7, 3 and 4:

$$\begin{pmatrix} (17,5) & (10,6) \\ (19,9) & (6,2) \end{pmatrix} \begin{pmatrix} (16,14) & (14,7) \\ (6,9) & (13,3) \end{pmatrix} \begin{pmatrix} (1,14) & (1,19) \\ (7,9) & (8,19) \end{pmatrix} \begin{pmatrix} (4,1) & (3,6) \\ (12,18) & (9,10) \end{pmatrix}$$

Payoffs of players 3 and 5, 3 and 6, 3 and 7, 4 and 5:

$$\begin{pmatrix} (9,9) & (10,7) \\ (13,6) & (19,2) \end{pmatrix} \begin{pmatrix} (14,15) & (16,12) \\ (5,13) & (8,13) \end{pmatrix} \begin{pmatrix} (2,12) & (1,8) \\ (10,6) & (14,5) \end{pmatrix} \begin{pmatrix} (15,12) & (6,14) \\ (12,6) & (4,2) \end{pmatrix}$$

Payoffs of players 4 and 6, 4 and 7, 5 and 6, 5 and 7, 6 and 7:

$$\begin{pmatrix} (3,12) & (9,11) \\ (14,7) & (3,15) \end{pmatrix} \begin{pmatrix} (19,19) & (12,1) \\ (12,4) & (2,6) \end{pmatrix} \begin{pmatrix} (16,6) & (8,7) \\ (2,16) & (19,2) \end{pmatrix}$$

$$\begin{pmatrix} (13,5) & (6,17) \\ (14,14) & (7,17) \end{pmatrix} \begin{pmatrix} (14,5) & (2,13) \\ (7,11) & (10,2) \end{pmatrix}$$

Firstly find the cooperative behaviour which maximizes the total sum of payoffs.

Table 1. Cooperative behaviour and payoffs from pairwise interaction

$\hat{\mathfrak{u}}_1^2 = B$	$\hat{\mathfrak{u}}_2^1 = B$	$\hat{\mathfrak{u}}_3^1 = A$	$\hat{\mathfrak{u}}_4^1 = A$	$\hat{\mathfrak{u}}_5^1 = B$	$\hat{\mathfrak{u}}_6^1 = B$	$\hat{\mathfrak{u}}_7^1 = B$
$h_1^2 = 20$	$h_2^1 = 5$	$h_3^1 = 1$	$h_4^1 = 11$	$h_5^1 = 17$	$h_6^1 = 16$	$h_7^1 = 20$
$\hat{\mathfrak{u}}_1^3 = B$	$\hat{\mathfrak{u}}_2^3 = B$	$\hat{\mathfrak{u}}_3^2 = A$	$\hat{\mathfrak{u}}_4^2 = A$	$\hat{\mathfrak{u}}_5^2 = A$	$\hat{\mathfrak{u}}_6^2 = A$	$\hat{\mathfrak{u}}_7^2 = B$
$h_1^3 = 13$	$h_2^3 = 8$	$h_3^2 = 15$	$h_4^2 = 20$	$h_5^2 = 9$	$h_6^2 = 14$	$h_7^2 = 19$
$\hat{\mathfrak{u}}_1^4 = B$	$\hat{\mathfrak{u}}_2^4 = B$	$\hat{\mathfrak{u}}_3^4 = B$	$\hat{\mathfrak{u}}_4^3 = A$	$\hat{\mathfrak{u}}_5^3 = B$	$\hat{\mathfrak{u}}_6^3 = A$	$\hat{\mathfrak{u}}_7^3 = B$
$h_1^4 = 11$	$h_2^4 = 14$	$h_3^4 = 12$	$h_4^3 = 18$	$h_5^3 = 2$	$h_6^3 = 15$	$h_7^3 = 5$
$\hat{\mathfrak{u}}_1^5 = B$	$\hat{\mathfrak{u}}_2^5 = B$	$\hat{\mathfrak{u}}_3^5 = B$	$\hat{\mathfrak{u}}_4^5 = A$	$\hat{\mathfrak{u}}_5^4 = A$	$\hat{\mathfrak{u}}_6^4 = A$	$\hat{\mathfrak{u}}_7^4 = A$
$h_1^5 = 7$	$h_2^5 = 19$	$h_3^5 = 19$	$h_4^5 = 15$	$h_5^4 = 12$	$h_6^4 = 7$	$h_7^4 = 19$
$\hat{\mathfrak{u}}_1^6 = A$	$\hat{\mathfrak{u}}_2^6 = A$	$\hat{\mathfrak{u}}_3^6 = A$	$\hat{\mathfrak{u}}_4^6 = B$	$\hat{\mathfrak{u}}_5^6 = A$	$\hat{\mathfrak{u}}_6^5 = A$	$\hat{\mathfrak{u}}_7^5 = B$
$h_1^6 = 6$	$h_2^6 = 16$	$h_3^6 = 14$	$h_4^6 = 14$	$h_5^6 = 16$	$h_6^5 = 6$	$h_7^5 = 14$
$\hat{\mathfrak{u}}_1^7 = A$	$\hat{\mathfrak{u}}_2^7 = B$	$\hat{\mathfrak{u}}_3^7 = B$	$\hat{\mathfrak{u}}_4^7 = A$	$\hat{\mathfrak{u}}_5^7 = B$	$\hat{\mathfrak{u}}_6^7 = A$	$\hat{\mathfrak{u}}_7^6 = A$
$h_1^7 = 12$	$h_2^7 = 8$	$h_3^7 = 14$	$h_4^7 = 19$	$h_5^7 = 14$	$h_6^7 = 14$	$h_7^6 = 5$

Now we need to calculate the values of the characteristic function for h-players and distribution for them.

$$V(H_1) = 25, \quad V(H_2) = 87, \quad V(H_3) = 70, \quad V(H_4) = 19, \quad V(\mathscr{H}) = 201,$$

$$K_{H_1} = 66.54, \quad K_{H_2} = 231.57, \quad K_{H_3} = 186.32, \quad K_{H_4} = 50.57.$$

On the second level we have three games on each hyperlink.
Game on H_1.

$$v(1) = v(2) = v(\emptyset) = 0, \quad v(1,2) = 25,$$
$$\phi_1^{H_1} = 12.5, \quad \phi_2^{H_1} = 12.5.$$

Game on H_2.

$$v(2) = v(3) = v(4) = v(\emptyset) = 0,$$
$$v(2,3) = 23, \quad v(2,4) = 34, \quad v(3,4) = 30, \quad v(2,3,4) = 87,$$
$$\phi_2^{H_2} = 28.5, \quad \phi_3^{H_2} = 26.5, \quad \phi_4^{H_2} = 32.$$

Game on H_3.

$$v(4) = v(5) = v(6) = v(\emptyset) = 0,$$
$$v(4,5) = 27, \quad v(4,6) = 21, \quad v(5,6) = 22, \quad v(4,5,6) = 87,$$
$$\phi_4^{H_3} = 24, \quad \phi_5^{H_3} = 24.5, \quad \phi_6^{H_3} = 21.5.$$

Game on H_4.

$$v(6) = v(7) = v(\emptyset) = 0, \quad v(6,7) = 19,$$
$$\phi_6^{H_3} = 9.5, \quad \phi_7^{H_3} = 9.5.$$

And the final step is to find the sum for each player.

$$T_1 = \frac{\phi_1^{H_1}}{\phi_1^{H_1} + \phi_2^{H_1}} K_{H_1} = 33.27,$$

$$T_2 = \frac{\phi_2^{H_1}}{\phi_1^{H_1} + \phi_2^{H_1}} K_{H_1} + \frac{\phi_2^{H_2}}{\phi_2^{H_2} + \phi_3^{H_2} + \phi_4^{H_2}} K_{H_2} = 109.12,$$

$$T_3 = \frac{\phi_3^{H_2}}{\phi_2^{H_2} + \phi_3^{H_2} + \phi_4^{H_2}} K_{H_2} = 70.53,$$

$$T_4 = \frac{\phi_4^{H_2}}{\phi_2^{H_2} + \phi_3^{H_2} + \phi_4^{H_2}} K_{H_2} + \frac{\phi_4^{H_3}}{\phi_4^{H_3} + \phi_5^{H_3} + \phi_6^{H_3}} K_{H_3} = 149.07,$$

$$T_5 = \frac{\phi_5^{H_3}}{\phi_4^{H_3} + \phi_5^{H_3} + \phi_6^{H_3}} K_{H_3} = 65.21,$$

$$T_6 = \frac{\phi_6^{H_3}}{\phi_4^{H_3} + \phi_5^{H_3} + \phi_6^{H_3}} K_{H_3} + \frac{\phi_6^{H_4}}{\phi_6^{H_4} + \phi_7^{H_4}} K_{H_4} = 82.515,$$

$$T_7 = \frac{\phi_7^{H_4}}{\phi_6^{H_4} + \phi_7^{H_4}} K_{H_4} = 25.285.$$

6. Conclusion

In the paper cooperative games with hypergraph communication structure was considered. Proposed two-level based approach for such class of games. Proved that the characteristic function for h-players is convex. The proposed solution divides the original game into subgames which reduced the complexity of calculations. Results were shown by the example.

References

Myerson R. B. (1977). *Graphs and cooperation in games.* Math. Oper. Res., **2**, 225–229.

Shapley, L. S. (1953). *A value for n-person games.* Annals of Math. Studies, **28**, 307–317.

Owen, G. (1986). *Values of graph-restricted games.* SIAM J. Alg. Disc. Meth., **7**, 210–220.

Meessen, R. (1988). *Communication games, Master's thesis.* Department of Mathematics. University of Nijmegen, the Netherlands (in Dutch).

van den Nouweland, A., Borm, P, Tijs, S. (1992). *Allocation rules for hypergraph communication situations.* Int. J. Game Theory, **20**, 255–268.

Shan, E., G. Zhang, X. Shan (2018). *The degree value for games with communication structure.* Int. J. Game Theory, **47**, 857–871.

Von Neumann, J., Morgenstern, O.(1994). Theory of Games and Economic Behavior. Princeton: Princeton University Press,

Bulgakova, M. A., Petrosyan, L. A. (2019). *About one multistage non-antagonistic network game.* Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes, **15(4)**, 603–615 (in Russian).

Tur, A., Petrosyan, L. (2021). *Strong Time-Consistent Solution for Cooperative Differential Games with Network Structure.* Mathematics, **9**, 755. https://doi.org/10.3390/math9070755